DIFFERENTIAL GRADED LIE ALGEBRA: A MOTIVATION

YUQING SHI

Up untill now we have been working with commutative differential graded algebras, *i.e.* the Sullivan model, for an algebraic description of rational spaces. Today we are going to motivate another "equivalent" (we will be more precise in the end) algebraic model for rational spaces, namely, differential graded Lie algebras, most of which is Quillen's contribution to rational homotopy theory, *cf.* [Qui69].

Notation.

i) Throughout the talk, $\mathbf{k}$ denotes a field of characteristic 0.

ii) If we omit the coefficients of chain complexes, homology groups and cohomology groups etc., we mean the corresponding objects with $\mathbf{k}$ coefficients.

1. Whitehead product and algebra structure of $\mathrm{H}_*(\Omega X)$

This section serves as a motivation for how Lie algebras appear naturally in algebraic invariants of spaces.

Situation 1.1. Let (X, x_0) be a simply connected, based topological space. Furthermore, assume $\mathrm{H}_*(X)$ is of finite type.

Let us first recall some facts about the based loop space of X:

i) The *based loop space* of X is defined as $\Omega X = \mathrm{Map}_{\mathrm{pt}}(\mathrm{S}^1, X)$.

ii) There is a fiber sequence $\Omega X \to PX \to X$. The associated long exact sequence of homotopy groups give the *connecting isomorphism* $\partial_* \colon \pi_*(X) \xrightarrow{\cong} \pi_{*-1}(\Omega X)$.

iii) There is an H-space structure $\mu\colon \Omega X \times \Omega X \to \Omega X$ on ΩX given by concatenation of loops. The induced maps on the singular chains $C_*^{\mathrm{sin}}(\mu)$ and on the homologies $H_*(\mu)$ makes $C_*^{\mathrm{sin}}(\Omega X)$ a chain algebra and $\mathrm{H}_*(\Omega X)$ a graded algebra.

Let us first introduce some algebraic definitions to make our statements precise.

Definition 1.2.

i) A *graded coalgebra* (over $\mathbf{k}$) C is a graded vector space equipped with linear maps
   i) (*comultiplication*) $\Delta\colon C \to C \otimes C$, and
   ii) (*counit*) $\epsilon\colon C \to \mathbf{k}$.

   of degree 0 such that
$$(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta$$
$$(\mathrm{id}_C \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{id}_C) \circ \Delta = \mathrm{id}_C .$$

ii) A *coaugmentation* of C is an inclusion $\eta\colon \mathbf{k} \to C$ of vector spaces such that
$$\epsilon(\eta(1)) = 1, \quad \Delta(\eta(1)) = \eta(1) \otimes \eta(1)$$
. Denote by 1 the element $\eta(1) \in C$.

iii) An element $c \in C$ is *primitive* if $\Delta(c) = 1 \otimes c + c \otimes 1$.

iv) The coalgebra C is *cocommutative* if $\tau \circ \Delta = \Delta$, where
$$\tau\colon C \otimes C \to C \otimes C,\ (a \otimes b) \mapsto (-1)^{\deg(a)\deg(b)} b \otimes a$$

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Remark 1.3.
  i) One should think about coalgebras as dual of algebras. The conditions for Δ and ϵ in Definition 1.2.i) are the dual versions of associativity and unitality of an algebra.
  ii) The primitive elements of C form a graded sub vector space of C, denoted by $P_*(C)$.

Example 1.4. Let Y be a topological space. The diagonal map $\Delta\colon Y \to Y \times Y$ induces a comultiplication on $C_*^{\mathrm{sin}}(Y)$ and $\mathrm{H}_*(Y)$. Thus the homology group $\mathrm{H}_*(Y)$ is a cocommutative coalgebra.

Definition 1.5. A *graded Hopf algebra* H is a graded vector space together with maps of graded vector spaces
  i) (multiplication) $\mu\colon H \otimes H \to H$,
  ii) (unit) $\eta\colon \mathbf{k} \to H$,
  iii) (comultiiplication) $\Delta\colon H \to H \otimes H$,
  iv) (counit) $\epsilon H \to \mathbf{k}$, and
  v) (antipode) $\chi\colon H \to H$,

such that (H, μ, η) is a graded algebra with augmentation ϵ, (H, Δ, ϵ) is a graded coalgebra with coaugmentation η, Δ is a morphism of graded algebras, and some condition satisfied by the antipode map.

Remark 1.6. In this talk, we will mainly focus on connected Hopf algebras. In this case, we can omit the antipode map from the above definition, *cf.* [MM65].

Example 1.7. Let X be a topological space satisfying Situation 1.1. Then the homology groups $\mathrm{H}_*(\Omega\, X)$ is a cocommutative graded Hopf algebra.

Let us first consider the primitive subspces of $\mathrm{H}_*(\Omega\, X)$

Notation 1.8. Let Y be a topological space. Denote by $P_*(Y) \coloneqq P_*(\mathrm{H}_*(Y))$ the primitive subspace of $\mathrm{H}_*(Y)$.

Exercise 1.9. Let $f\colon Y \to Z$ be a map of topological spaces. Show that the restriction of $\mathrm{H}_*(f)$ on the primitive subspace $P_*(Y)$ of $\mathrm{H}_*(Y)$ maps to the primitive subspace $P_*(Z)$ of $\mathrm{H}_*(Z)$.

Corollary 1.10. *Let Y be a topological space. The image of the Hurewicz map is contained in $P_*(Y)$.*

Sketch. For $k \geq 0$, the generator of $\mathrm{H}_*(\mathrm{S}^k)$ is primitive. □

Theorem 1.11 (Cartan–Serre)**.** *Let X be a topological space satisfying Situation 1.1. The Hurewicz map of $\Omega\, X$ gives an isomorphism*

$$\pi_*(\Omega\, X) \otimes \mathbf{k} \xrightarrow{\cong} P_*(\Omega\, X).$$

Sketch. Let $\big\{\{g_i\colon \mathrm{S}^{2l_i+2} \to X\}_{i\in I},\ \{u_j\colon \mathrm{S}^{2k_j+1} \to X\}_{j\in J}\big\}$ represent a basis of $\pi_*(X) \otimes \mathbf{k}$. For $i \in I$, there are induced maps $\overline{g_i}\colon \mathrm{S}^{2l_i+1} \to \Omega\, X$ such that $[\overline{g_i}] = \partial_*([g_i])$ (∂_* is the connecting isomorphism). For $j \in J$, there are induced maps $\Omega\, u_j\colon \Omega\, \mathrm{S}^{2k_j+1} \to \Omega\, X$. Thus we can define a map

$$f = (\overline{g_i})_{i\in I} \times (\Omega\, u_j)_{j\in J} \colon \widetilde{\prod}_{i\in I} \mathrm{S}^{2l_i+1} \times \widetilde{\prod}_{j\in J} \Omega\, \mathrm{S}^{2k_j+1} \to \Omega\, X,$$

where $\widetilde{\prod}$ denotes the weak product, *cf.* [FHT01, Chapter 16.c]. In fact, $\pi_*(f) \otimes \mathbf{k}$ and $\mathrm{H}_*(f; \mathbf{k})$ are isomorphisms, *cf.* [FHT01, Proposition 16.7].

To finish the proof, one need to check that the Hurewicz map for S^{2l_i+1} and $\Omega\,\mathrm{S}^{2k_j+1}$ are isomorphisms into the primitive subspaces, see [FHT01, Proposition 16.7] for more details. □

Remark 1.12. The sketch above tells us that a loop space is rational homotopy equivalent to weak product of odd spheres and loop spaces of odd spheres.

Actually, the algebraic structure of $\mathrm{H}_*(\Omega\, X)$ can be completely determined by $\pi_*(X)\otimes\mathbf{k}$.

Definition 1.13. Let Y be a topological space.

i) Let $\alpha\in\pi_k(Y)$ and $\beta\in\pi_l(Y)$. Define the *Whitehead product* $[\alpha,\beta]_W\in\pi_{k+l-1}(Y)$ as the homotopy class of the following map

$$\mathrm{S}^{k+l-1}\xrightarrow{m}\mathrm{S}^k\times\mathrm{S}^l\xrightarrow{\alpha\vee\beta}Y$$

where m is the attaching map of the top cell of $\mathrm{S}^k\times\mathrm{S}^l$.

ii) Define an induced bracket $[-,-]$ on $\pi_*(\Omega\,Y)$ via the connecting homomorphism ∂_*

$$[\partial_*\alpha,\partial_*\beta]:=(-1)^{\deg\alpha}\partial_*([\alpha,\beta]_W).$$

Denote the graded vector space $\pi_*(\Omega\,Y)\otimes\mathbf{k}$ with the bracket $[-,-]$ by L_Y.

Theorem 1.14. *The map* $\theta\colon\pi_*(Y)\otimes\mathbf{k}\xrightarrow{\partial_*}\pi_{*-1}(\Omega\,Y)\otimes\mathbf{k}\xrightarrow{h_*}\mathrm{H}_*(\Omega\,Y)$ *extends to an isomorphism*

$$TL_Y/I\xrightarrow{\cong}\mathrm{H}_*(\Omega\,Y),$$

of graded algebras, where TL_Y *denotes the tensor algebra of* L_Y *and* I *is the ideal generated by* $\alpha'\otimes\beta'-(-1)^{\deg(\alpha')\deg(\beta')}\beta'\otimes\alpha'-[\alpha',\beta']$ *for* $\alpha',\beta'\in\pi_*(\Omega\,Y)\otimes\mathbf{k}$.

There is an algebraic analog of Theorem 1.14.

2. Graded Lie algebra

Definition 2.1.

i) A *graded Lie algebra* (over $\mathbf{k}$) L is a graded vector space $L=\{L_i\}_{i\in\mathbb{Z}}$ together with a linear map $[-,-]\colon L\otimes L\to L$ of degree 0, called *Lie bracket*, such that
   i) (antisymmetry) for any $x,y\in L$, we have $[x,y]=-(-1)^{\deg(x)\deg(y)}[y,x]$, and
   ii) (Jacobi identity) for any $x,y,z\in L$, we have

$$[x,[y,z]]=[[x,y],z]+(-1)^{\deg(x)\deg(y)}[y,[x,z]].$$

ii) A *morphism of graded Lie algebras* is linear map of degree zero that preserves the Lie bracket.

Example 2.2. Let Y be a topological space. Then L_Y is a graded lie algebra.

Example 2.3. Given a graded algebra A, we can define a graded Lie algebra A^{Lie} via the *commutator bracket* $[a_1,a_2]:=a_1a_2-(-1)^{\deg a_1\deg(a_2)}a_2a_1$. This assignment $(-)^{\mathrm{Lie}}$ is functorial and admits a left adjoint

$$U:\mathbf{grLieAlg}\rightleftarrows\mathbf{grAlg}:(-)^{\mathrm{Lie}}.$$

For a graded Lie algebra L, we say UL is the *universal enveloping algebra* associated to L. The graded algebra UL has the following universal property: for any graded algebra

A and a morhpism of graded Lie algebras $L \to A^{\mathrm{Lie}}$, there exists a unique graded algebra morphism $\phi\colon UL \to A$ such that the following diagram commutes

$$\begin{tikzcd} L \arrow[r] \arrow[dr] & UL^{\mathrm{Lie}} \arrow[d, "\phi"] \\ & A^{\mathrm{Lie}} \end{tikzcd}$$

The universal enveloping algebra UL has the following properties:

i) There is an graded algebra isomorphism $UL \cong TL/I$ where I is the ideal generated by elements of the form $x \otimes y - (-1)^{\deg(x)\deg(y)} y \otimes x - [x, y]$.

ii) The diagonal map $\Delta\colon L \to L \otimes L$ and the map $z\colon L \to \mathbf{k}, l \mapsto 0$ induce a cocommutative Hopf algebra structure on UL, where the comultiplication and counit is given by $U(\Delta)$ and $U(z)$.

Example 2.4. Let H be a graded Hopf algebra. The primitive subspace $P_*(H)$ of H is a graded Lie algebra with the commutator bracket.

Theorem 2.5 (Milnor-Moore)**.** *The functor U and P gives an equivalence between the category of connected graded Lie algebras and the category of connected cocommutative graded Hopf algebras.*

Corollary 2.6. *We have an isomorphism $L \cong P_*(UL)$ of graded Lie algebras, for L a connected graded Lie algebra.*

The following corollary is a reinterpretation of Theorem 1.14.

Corollary 2.7. *Let Y be a topological space. There is an isomorphism*

$$UL_Y \cong \mathrm{H}_*(\Omega Y)$$

of graded Hopf algebras.

3. Quillen's idea

Quillen's idea is to have a "chain level" statement of the Corollary 2.7. More precisely:

Definition 3.1. A *differential graded Lie algebra* is a graded Lie algebra equipped with differentials $\{d_i\}_{i\in\mathbb{Z}}$

$$\cdots \to L_{n+1} \xrightarrow{d_{n+1}} L_n \xrightarrow{d_n} L_{n-1} \to \cdots$$

such that $d[x, y] = [dx, y] + (-1)^{\deg(x)}[x, dy]$.

Remark 3.2. Similar as in Section 2. we can define the differential universal enveloping algebra associated to a differential graded Lie algebra, and differential graded Hopf algebra. We also have the adjunction from Example 2.3 between differential graded Lie algebra and differential graded algebras and Theorem 2.5 is also true in differential graded setting.

For each simply connected space Y, Quillen's idea is to find a differential graded Lie algebra $\lambda(X)$ such that:

i) There is a natural isomorphism of graded Lie algebras $\mathrm{H}_*(\lambda(X)) \cong L_X$.

ii) There is an equivalence $C_*^{\sin}(\Omega X) \simeq U(\lambda(X))$ of differential graded Hopf algebra.

The second point can not be true because $C_*^{\sin}(\Omega X)$ is only cocommutative up to homotopy and $U(\lambda(X))$ is cocommutative. Nevertheless, Quillen has the following important theorem:

Theorem 3.3. *There are Quillen equivalences*

$$\mathbf{Top}^{\mathbb{Q}}_{\geq 2} \rightleftarrows \mathbf{dgla}^{\mathbb{Q}}_{\geq 1} \rightleftarrows \mathbf{dgC}^{\mathbb{Q}}_{\geq 2},$$

where the categories from left to right are the category of simply connected rational topological space, the category of connected differential graded Lie algebra over $\mathbb{Q}$ and the category of simply connected differential graded coalgebras over $\mathbb{Q}$.

Remark 3.4.

i) The category $\mathbf{dgla}^{\mathbb{Q}}_{\geq 1}$ and $\mathbf{dgC}^{\mathbb{Q}}_{\geq 2}$ are algebraic models for simply connected rational topological spaces.

ii) The category of differential graded coalgebras of finite type is equivalent to the category of differential graded algebras of finite type by taking the dual. Thus we see that the algebraic models are equivalent, if we restrict to the subcategory of simply connected rational spaces whose homology groups are of finite type.

To end, let us introduce a theorem which shows that rational homotopy theory is an unstable homotopy theory. In other words, a lot of information are "lost" when one applies the suspension or the loop functor.

Theorem 3.5. *Let X be a space that satisfies Situation 1.1. The following statements are equivalent:*

i) *The space X is rational equivalent to a wedge of Moore spaces.*

ii) *The space X is rational equivalent to suspension of some space.*

iii) *The rational Hurewicz maps is surjective in every degree.*

iv) *The Lie algebra L_X is a free differential graded Lie algebra*

Proof. See [FHT01, Theorem 24.5] □

References

[FHT01] Yves Felix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory.*, volume 205. New York, NY: Springer, 2001.

[MM65] John W. Milnor and J. C. Moore. On the structure of Hopf algebras. *Ann. Math. (2)*, 81:211–264, 1965.

[Qui69] Daniel Quillen. Rational homotopy theory. *Ann. Math. (2)*, 90:205–295, 1969.