RELATIVE SULLIVAN ALGEBRA AND MODELS OF FIBRATIONS

YUQING SHI

This is the notes of my talk in rational homotopy theory seminar, ran by Gijs Heuts and Lennart Meier in Spring 2019 at Utrecht geometry centre. In this talk we will introduce the relative Sullivan algebra and use this to construct Sullivan models for fibrations.

Notation. Through out the talk

i) we fix the notation k as a field of characteristic 0,
ii) we denote quasi-isomorphism by $\simeq$ and isomorphism by $\cong$, and
iii) we assume that all topological spaces are path-connected.

1. Basics about relative Sullivan algebras

Definition 1.0.1. A *relative Sullivan algebra* is a cdga of the form $(B \otimes \Lambda V, d)$ where

i) (B, d) is a cdga with $\mathrm{H}^0(B) \cong k$,
ii) $V = \{V^p\}_{p\geq 1}$ is a graded vector space, and
iii) there is an increasing sequence $V(0) \subsetneqq V(1) \subsetneqq \cdots$ of subspaces of V such that $V = \bigcup_{k=0}^{\infty} V(n)$ and

$$d\colon V(0) \to B \text{ and } d\colon V(n) \to B \otimes \Lambda V(n-1) \text{ for } k \geq 1.$$

The cdga (B, d) is called the *base algebra* of $(B \otimes \Lambda V, d)$.

Remark 1.0.2.

i) We can embed B and V in $(B \otimes \Lambda V, d)$ as $B \otimes 1$ and $1 \otimes V$ respectively.
ii) The condition on the differential d in iii) in Definition 1.0.1 is called the *nilpotence condition*. Let us define $V(-1) := 0$ and sub vector spaces $V_n \subseteq V$ such that $V(n) = V(n-1) \oplus V_n$. Then the nilpotence condition is equivalent to

$$d\colon V_n \to B \otimes \Lambda V(n-1), \text{ for } n \geq 0.$$

Definition 1.0.3. A *Sullivan algebra* is a relative Sullivan algebra with $B = k$.

Definition 1.0.4. Let $(B \otimes \Lambda V, d)$ and $(B' \otimes \Lambda V', d)$ be two relative Sullivan algebras with base algebras B and B' respectively. We say that a morphism $\eta\colon (B \otimes \Lambda V, d) \to (B' \otimes \Lambda V', d)$ of cdga's is a *morphism of relative Sullivan algebras* if we have the following commutative diagram

$$\begin{array}{ccc} (B, d) & \xrightarrow{\eta|_B} & (B', d) \\ {\scriptstyle i_B}\downarrow & & \downarrow{\scriptstyle i_{B'}} \\ (B \otimes \Lambda V, d) & \xrightarrow[\eta]{} & (B' \otimes \Lambda V', d). \end{array}$$

where i_B and $i_{\widetilde{B}}$ are inclusions of the base algebras.

Remark 1.0.5. One can also think about relative Sullivan algebras as analogue to CW-pairs. The cdga $(B \otimes \Lambda V, d)$ can be obtained from (B, d) by "attaching disks algebras" $D(n-1)$ along "sphere algebras" $S(n)$ for $n \in \mathbb{N}$. For more rigorous explanation of this, see [Ber12, Definition 8.3, Example 8.4].

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Example 1.0.6.

i) Any cdga $(\Lambda V, d)$ with $V = V^{\geq 2}$ and $\operatorname{im} d \subseteq \Lambda^+ V \otimes \Lambda^+ V$ is a (minimal) Sullivan algebra. Here we can consider $V(n) := \oplus_{i=2}^n V^n$.

ii) (Sullivan fibre) Let $(B \otimes \Lambda V, d)$ be a relative Sullivan algebra and $\epsilon\colon (B, d) \to k$ be an augmentation. We obtain a Sullivan algebra $(\Lambda V, \overline{d}) \cong k \otimes_B (B \otimes \Lambda V, d)$ via the pushout diagram

$$\begin{array}{ccc} (B,d) & \xrightarrow{\epsilon} & k \\ {\scriptstyle i}\downarrow & & \downarrow \\ (B \otimes \Lambda V, d) & \xrightarrow[\epsilon \otimes \mathrm{id}]{} & k \otimes_B (B \otimes \Lambda V, d) \end{array}$$

We call $(\Lambda V, \overline{d})$ the *Sullivan fibre at* ϵ. We will see in Section 5 why it is called "fibre".

Exercise 1.0.7. Show that the cdga $(\Lambda(v_1, v_2, v_3), d)$ with $\deg v_i = 1$ and $dv_1 = v_2 \otimes v_3$, $dv_2 = v_1 \otimes v_3$ and $dv_3 = v_1 \otimes v_2$ is not a Sullivan algebra.

2. Sullivan models

In this section we will introduce the Sullivan models for certain morphisms of cdga's. Compared with minimal models that are introduced in previous talks, we can eliminate the 1-reduced condition if we work with Sullivan models.

Situation 2.0.1. For this section, fix $\phi\colon (B, d) \to (C, d)$ to be a morphism of cdga's with $\mathrm{H}^0(C) \cong k$.

Definition 2.0.2.

i) A *Sullivan model* for ϕ is a quasi-isomorphism $m\colon (B \otimes \Lambda V, d) \xrightarrow{\simeq} (C, d)$ of cdga's such that $(B \otimes \Lambda V, d)$ is a relative Sullivan algebra and the following diagram commutes.

$$\begin{array}{ccc} (B,d) & \xrightarrow{\phi} & (C,d) \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle m}\,{\scriptstyle \simeq} & \\ & (B \otimes \Lambda V, d) & \end{array}$$

ii) In the case $B = k$, the Sullivan model $(\Lambda V, d)$ is called a Sullivan model for (C, d).

iii) Let $f\colon X \to Y$ be a morphism of topological spaces. A Sullivan model for f is a Sullivan model for $\mathrm{A}_{\mathrm{PL}}(f)$. We also call a Sullivan model for $\mathrm{A}_{\mathrm{PL}}(X)$ a Sullivan model for X, for a topological space X.

Now let us consider the existence of a Sullivan model for ϕ.

Proposition 2.0.3. *A morphism ϕ of cdga's as in Situation 2.0.1 has a Sullivan model if and only if the induced map $\mathrm{H}^0(\phi)$ is an isomorphism and $\mathrm{H}^1(\phi)$ is injective.*

We need to introduce some properties of relative Sullivan algebras before we can prove this proposition.

Definition 2.0.4. Let (R, d) be a cdga and (M, d) be a module of (R, d). We say (M, d) is *semifree* if M is the union of an increasing sequence $M(0) \subsetneqq M(1) \subsetneqq \cdots$ of submodules of (M, d) such that $M(0)$ and $M(n)/M(n-1)$, for $n \geq 1$, are (R, d)-free.

Lemma 2.0.5. *Let $(B \otimes \Lambda V, d)$ be a relative Sullivan algebra. The multiplication on B makes $(B \otimes \Lambda V, d)$ a (B, d)-module. Furthermore, $(B \otimes \Lambda V, d)$ is (B, d)-semifree.*

Proof. See [FHT01, Lemma 14.1] for a proof. □

Lemma 2.0.6. *Let $(B \otimes \Lambda V, d)$ be a relative Sullivan algebra. Let $\psi\colon (B, d) \to (B', d)$ be a morphism of cdga's with $\mathrm{H}^0(B') \cong k$. Then the cdga $(B' \otimes \Lambda V, d)$ obtained by the pushout diagram*

$$\begin{array}{ccc} (B,d) & \xrightarrow{\ i\ } & (B \otimes \Lambda V, d) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f\otimes \mathrm{id}} \\ & \ulcorner & \\ (B',d) & \longrightarrow & (B' \otimes \Lambda V, d) \end{array}$$

is a relative Sullivan algebra with base (B', d).

Proof. See [FHT01, Section 14.1] for a proof. □

Lemma 2.0.7. *If f from Lemma 2.0.6 is a quasi-isomorphism, then $f \otimes \mathrm{id}$ is an quasi-isomorphism.*

Proof. See [FHT01, Lemma 14.2] for a proof. □

Remark 2.0.8. The proof of Lemma 2.0.7 uses the concept of semifreeness.

The idea of the proof of Proposition 2.0.3 is to first prove the existence of Sullivan model for a "simpler" subcdga $\widetilde{B}$ of B, and then use Lemma 2.0.7 to prove the existence for B.

Sketch of proof of Proposition 2.0.3. For the "⇒" direction, the proof follows by direct computation.

For the "⇐" direction, let us choose a sub cdga $\widetilde{B} \subseteq B$ such that $\widetilde{B}^0 = k$, $\widetilde{B} \oplus d(B^0) = B^1$ and $\widetilde{B}^n = B^n$ for $n \geq 2$. Then we see that the inclusion $j\colon (\widetilde{B}, d) \to (B, d)$ is an quasi-isomorphism. Denote $\widetilde{\phi} \coloneqq \phi|_{\widetilde{B}}$. We have that $\mathrm{H}^1(\widetilde{\phi})$ is injective.

Claim 2.0.9. *The map $\widetilde{\phi}$ has a Sullivan model $\widetilde{m}\colon (\widetilde{B} \otimes \Lambda V, d) \xrightarrow{\simeq} (C, d)$.*

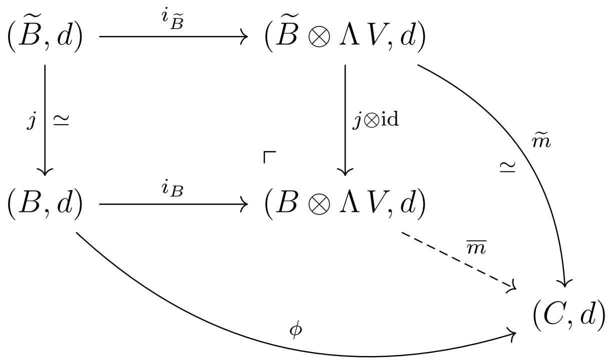
The idea of the proof of this claim is to construct graded vector spaces V_n, extend the differential of $\widetilde{B}$ to $\widetilde{B} \otimes \Lambda(\oplus_{i=1}^n V_i)$ and define maps $\widetilde{m}_n\colon \widetilde{B} \otimes \Lambda(\oplus_{i=1}^n V_i) \to (C, d)$ for $n = 0, 1, \ldots$ inductively.

Sketch for Claim 2.0.9. We will sketch the constructions for the base case and leave the rest of the proof as an exercise. Let $\{b_m\}$ be a set of cocycles in $\widetilde{B}$ such that $\{[b_m]\}$ is a basis for $\ker(\mathrm{H}(\widetilde{\phi}))$. We define the graded vector space $W_1 \coloneqq \mathrm{span}\langle w_m \rangle$ where the collection $\{w_m\}$ is in bijection with $\{b_m\}$ and define $\deg(w_m) \coloneqq \deg(b_m) - 1$. First, we extend the differential of $\widetilde{B}$ to $\widetilde{B} \otimes \Lambda W_1$ by defining $d(w_m) = b_m$. Then, we can extend the map $\widetilde{\phi}$ to $m_{W_1}\colon \widetilde{B} \otimes W_1 \to (C, d)$ by setting $m_{W_1}(w_m) \coloneqq c_m$ for some c_m in C such that $\phi(b_m) = d(c_m)$. Choose a vector space W_2 such that $W_2 \cong \mathrm{H}^+(C)/\mathrm{im}(\mathrm{H}^+(\phi'))$ and define $V_0 \coloneqq W_1 \oplus W_2$. Then we can extend d to $\widetilde{B} \otimes \Lambda V_0$ by setting $d|_{W_1} = 0$, and we can extend m_{W_1} to a map $m_0\colon \widetilde{B} \otimes \Lambda V_0$ by sending a basis of W_2 to the basis of $\mathrm{H}^+(C)/\mathrm{im}(\mathrm{H}^+(\phi'))$ under the chosen isomorphism.

Exercise 2.0.10. Complete the proof of Claim 2.0.9.

□

To show the existence of a Sullivan model for ϕ, consider the pushout diagram



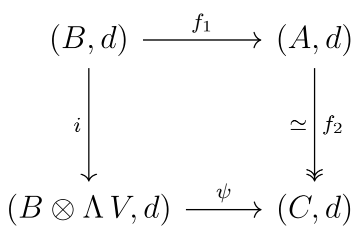
Since the map j is an quasi-isomorphism, we have that the map $j \otimes \mathrm{id}$ is an quasi-isomorphism by Lemma 2.0.7. Therefore by commutativity of the diagrams, we have that $\overline{m}$ is a quasi-isomorphism and $\overline{m}$ is a Sullivan model for ϕ. □

3. Homotopies of the relative Sullivan algebras

In this short section we will define homotopies between maps from a relative Sullivan algebra and introduce some lifting properties of relative Sullivan algebras. We will use these properties in the next section to prove the existence and uniqueness of the minimal Sullivan models for a morphism of cdga's.

First, let us see a lifting property of the inclusion $i\colon (B, d) \to (B \otimes \Lambda V, d)$ of the base algebra into a relative Sullivan algebra with respect to a surjective quasi-isomorphism of cdga's.

Lemma 3.0.1. *Given a commutative diagram of cdga's*



where $(B \otimes \Lambda V, d)$ is a Sullivan algebra with inclusion i of its base (B, d) and f_2 is a surjective quasi-isomorphism. Then there exists a morphism $\psi'\colon (B \otimes \Lambda V, d) \to (A, d)$ of cdga's such that $\psi' \circ i = f_1$ and $f_2 \circ \psi' = \psi$.

Proof. See [FHT01, Lemma 14.4] for a proof. □

Now let us define the homotopy of maps from a relative Sullivan algebra.

Definition 3.0.2. Let $\psi_0, \psi_1\colon (B \otimes \Lambda V, d) \to (A, d)$ be morphisms of cdga's where $(B \otimes \Lambda V, d)$ is relative Sullivan algebra and $f \coloneqq \psi_0|_B = \psi_1|_B$. We say ψ_0 and ψ_1 are homotopic rel B, write $\psi_0 \sim_B \psi_1$, if there is a morphism

$$\varPsi\colon (B \otimes \Lambda V, d) \to (A, d) \otimes (\Lambda(t, dt), d)$$

of cdga's such that $(\mathrm{id}_A \otimes \epsilon_0) \circ \varPsi = \psi_0$ and $(\mathrm{id}_A \otimes \epsilon_1) \circ \varPsi = \psi_1$ and $(\mathrm{id}_A \otimes \epsilon_n) \circ \varPsi|_B = f$ for all $n \in k$. Here, the $\epsilon_n\colon (\Lambda(t, dt), d) \to k$ denotes the augmentation sending t to n. We say the map $\varPsi$ a *homotopy rel B* between ψ_0 and ψ_1.

Remark 3.0.3. This is an equivalence relation on the collection of maps of cdga's whose domain is a relative Sullivan algebra and the maps agrees on the restrictions on the base algebra, *cf.* [FHT01, Proposition 12.7].

Now we can introduce another lifting property of the inclusion $i\colon (B,d) \to (B \otimes \Lambda V, d)$ of the base algebra into a relative Sullivan algebra.

Proposition 3.0.4. *Given a commutative diagram of cdga's*

$$\begin{array}{ccc} (B,d) & \xrightarrow{f_1} & (A,d) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \simeq\, f_2} \\ (B\otimes \Lambda V, d) & \xrightarrow[\psi]{} & (C,d) \end{array}$$

where $(B \otimes \Lambda V, d)$ is a Sullivan algebra with inclusion i of its base (B,d) and f is a quasi-isomorphism (not necessarily surjective). Then there is a morphism $\psi'\colon (B \otimes \Lambda V, d) \to (A,d)$, unique *up to homotopy rel B, such that $\psi' \circ i = f_1$ and $f_2 \circ \psi' \sim_B \psi$.*

Proof. See [FHT01, Proposition 14.6] for a proof. □

4. Minimal Sullivan models

Definition 4.0.1. A relative Sullivan algebra $(B \otimes \Lambda V, d)$ is *minimal* if

$$\operatorname{im} d \subseteq B^+ \otimes \Lambda V + B \otimes \Lambda^{\geq 2} V.$$

The aim of this section is to show the existence and uniqueness of minimal Sullivan models for certain morphisms of cdga's. This summarised into the following theorem.

Theorem 4.0.2. *Let $\phi\colon (B,d) \to (C,d)$ be a morphism of cdga's such that the induced map $\mathrm{H}^0(\phi)$ is isomorphism and $\mathrm{H}^1(\phi)$ is injective. Then*

i) *the map ϕ has a minimal Sullivan model $m\colon (B \otimes \Lambda V, d) \xrightarrow{\simeq} (C,d)$.*
ii) *Let $m'\colon (B \otimes \Lambda V', d) \xrightarrow{\simeq} (C,d)$ be a second minimal Sullivan model for ϕ. Then there exists an isomorphism $\eta\colon (B \otimes \Lambda V, d) \to (B \otimes \Lambda V', d)$ of relative Sullivan algebras restricting to id_B such that $m' \circ \eta \sim_B m$.*

Corollary 4.0.3. *Any cdga (A,d) with $\mathrm{H}^0(A) \cong k$ and any path-connected topological space X have a minimal Sullivan model, unique up to isomorphisms.*

We need to introduce some theorems first before we can prove Theorem 4.0.2.

Theorem 4.0.4. *Let $(B \otimes \Lambda V, d)$ be a relative Sullivan algebra. Then there exists an isomorphism*

$$f\colon (B \otimes \Lambda W, d) \otimes (\Lambda(U \oplus dU), d) \xrightarrow{\cong} (B \otimes \Lambda V, d)$$

of cdga's where $(B \otimes \Lambda W, d)$ is a minimal Sullivan algebra and $(\Lambda(U \oplus dU), d)$ is contractable.

Proof. See [FHT01, Theorem 14.9] for a proof. □

Theorem 4.0.5. *Let $\eta\colon (B \otimes \Lambda V, d) \xrightarrow{\simeq} (B' \otimes \Lambda V, d)$ be a quasi-isomorphism of minimal relative Sullivan algebra. If $\eta|_B : (B,d) \to (B',d)$ is an isomorphism of cdga's, then η is an isomorphism.*

Proof. See [FHT01, Theorem 14.11] for a proof. □

Proof of Theorem 4.0.2. By Proposition 2.0.3 we know that ϕ has a Sulliva model $m\colon (B \otimes \Lambda V, d) \to (C,d)$. Using Theorem 4.0.4 we can assume that $(B \otimes \Lambda V, d)$ is minimal. This proves the existence.

For uniqueness let us take a look at the following commutative diagram of cdga's

$$\begin{array}{ccc} (B,d) & \xrightarrow{i'} & (B\otimes \Lambda V',d) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \simeq\, m'} \\ (B\otimes \Lambda V,d) & \xrightarrow[m]{\simeq} & (C,d) \end{array}$$

By Proposition 3.0.4 we obtain a morphism $\eta\colon (B\otimes \Lambda V,d)\to(B\otimes\Lambda V',d)$ of cdga's such that $m'\circ\eta\sim_B m$ and $\eta\circ i=i'$. Thus η is a morphism of relative Sullivan algebras. By (homotopy) commutativity of the diagrams we have $\eta|_B=\mathrm{id}_B$ and η is a quasi-isomorphism. Therefore η is an isomorphism by Theorem 4.0.5. □

5. Models of fibrations

In the last section of the talk we will introduce an application of Sullivan algebras. Using a Sullivan model of a Serre fibrations, we can obtain a Sullivan model for the fibre of a Serre fibration, and hence (Remark 5.0.8) a model for the homotopy fibre of a continuous map.

Let $p\colon X\to Y$ be a Serre fibration with F.

We know that we can obtain F from the pullback diagram

$$\begin{array}{ccc} F & \xrightarrow{i_F} & X \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \{y_0\} & \xrightarrow[i_{y_0}]{} & Y \end{array}$$

where y_0 is a base point of Y such that $p(F)=y_0$. Applying the functor $\mathrm{A_{PL}}(-)$ to this diagram, we have

$$\begin{array}{ccc} \mathrm{A_{PL}}(F) & \xleftarrow{\mathrm{A_{PL}}(i_F)} & \mathrm{A_{PL}}(X) \\ {\scriptstyle \mathrm{A_{PL}}(p)}\uparrow & & \uparrow{\scriptstyle \mathrm{A_{PL}}(p)} \\ k & \xleftarrow[\epsilon]{} & \mathrm{A_{PL}}(Y) \end{array} \tag{5.0.1}$$

Exercise 5.0.1. Show that $\mathrm{H}^1(\mathrm{A_{PL}}(p))$ is injective.

Therefore there exists a Sullivan model $m\colon(\mathrm{A_{PL}}(Y)\otimes\Lambda V,d)\to(\mathrm{A_{PL}}(X))$ for p. Recall the definition of Sullivan fibre of $(\mathrm{A_{PL}}(Y)\otimes\Lambda V,d)$ at the augmentation ϵ, *cf.* Example 1.0.6.ii), we obtain the following commutative diagrams of cdga's

$$\begin{array}{ccccc} \mathrm{A_{PL}}(Y) & \xrightarrow{\epsilon} & k & & \\ {\scriptstyle i}\downarrow & & \downarrow & \searrow{\scriptstyle \mathrm{A_{PL}}(p)} & \\ (\mathrm{A_{PL}}(Y)\otimes\Lambda V,d) & \xrightarrow{\epsilon\otimes\mathrm{id}} & (\Lambda V,\bar d) & & \\ \downarrow{\scriptstyle m} & & & \overset{\bar m}{\searrow} & \\ \mathrm{A_{PL}}(X) & & \xrightarrow{\mathrm{A_{PL}}(i_F)} & & \mathrm{A_{PL}}(F) \end{array} \tag{5.0.2}$$

Situation 5.0.2. Let Y be simply connected and one of $\mathrm{H}_*(Y;k)$ and $\mathrm{H}_*(X;k)$, considered as graded vector spaces, is of finite type.

Theorem 5.0.3. *Let $p\colon X \to Y$ be Serre fibration of path-connected spaces with path-connected fibre F that satisfies the properties in Situation 5.0.2. Then the morphism $\overline{m}$ in the diagram 5.0.2 is a quasi-isomorphism. In other words, the Sullivan fibre $(\Lambda V, \overline{d})$ is a Sullivan model of the fibre of p.*

Proof. We will prove the theorem in the case where p is a fibration and leave the proof of the case where p is a Serre fibration as an exercise.

Claim 5.0.4. *The theorem is true if p is a fibration.*

Proof. We will prove first a more general statement.

Theorem 5.0.5 (Eilenberg-Moore)**.** *Let D be a commutative diagram of cochain complexes over k:*

(D)
$$\begin{array}{ccc} A & \xleftarrow{\;f\;} & E \\ \uparrow & & \uparrow \\ k & \xleftarrow[\epsilon]{} & B. \end{array}$$

By choosing a B-semifree module M_B with an quasi-isomorphism $m_B\colon M_B \xrightarrow{\simeq} E$, we obtain another commutative diagram D'

(D')
$$\begin{array}{ccc} A & \xleftarrow{\;f\;} & E \\ {\scriptstyle \overline{m}_B}\uparrow & & \uparrow{\scriptstyle \simeq\, m_B} \\ k \otimes_B M_B & \xleftarrow[\epsilon]{} & B \otimes_B M_B. \end{array}$$

If D is weakly equivalent to a third commutative diagram of chain complexes with k-coefficients

(D_p)
$$\begin{array}{ccc} C^*(F) & \xleftarrow{C^*(i_F)} & C^*(Y) \\ \uparrow & & \uparrow{\scriptstyle C^*(p)} \\ k & \xleftarrow[\epsilon]{} & C^*(Y). \end{array}$$

where $C^*(-)$ is the singular cochain functor and $p\colon X \to Y$ is a fibration that satisfies properties in Situation 5.0.2 and F is its fibre, then the map $\overline{m}_B$ is a quasi-isomorphism.

For the proof of the claim, let D be the diagram 5.0.1 and let M_B be the relative Sullivan algebra $(\mathrm{A_{PL}}(Y) \otimes \Lambda V, d)$. Then the diagram D' is

$$\begin{array}{ccc} \mathrm{A_{PL}}(F) & \xleftarrow{\mathrm{A_{PL}}(i_F)} & \mathrm{A_{PL}}(X) \\ {\scriptstyle \overline{m}}\uparrow & & \uparrow{\scriptstyle m} \\ (\Lambda V, \overline{d}) & \xleftarrow[\epsilon]{} & (\mathrm{A_{PL}}(Y) \otimes \Lambda V, d). \end{array}$$

The diagram D is weakly equivalent to D_p because $C^*(X)$ and $A_{PL}(X)$ are natural quasi-isomorphic for any topological space X. Thus we conclude by Theorem 5.0.5 that $\overline{m}$ is a quasi-isomorphism. □

The idea of the proof in the case where p is a Serre fibration is to replace p by a fibration. We leave this as an exercise.

Exercise 5.0.6. Prove the theorem for p is a Serre fibration.

□

Remark 5.0.7. In Theorem 5.0.5, M_B is called a *B-semifree resolution* of E. For the proof of existence of semifree resolution, see [FHT01, Proposition 6.6]. Also, see [FHT01, Theorem 7.10] for a proof of Theorem 5.0.5

Remark 5.0.8. Using the proof idea of Exercise 5.0.6, we can see that the theorem is also true when p is a continuous map of topological spaces. In this case, we see that $(\Lambda V, \overline{d})$ is a Sullivan model for the homotopy fiber of p.

References

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[FHT01] Y. Felix, S. Halperin, and J.-C. Thomas. *Rational homotopy theory.* Vol. 205. New York, NY: Springer, 2001,