PROOF OF [DHS88, STEP II]

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In this talk, we would like to prove the following

Theorem 0.1 ([DHS88, Step II]). *Let R be a connective commutative ring spectrum and $\alpha \in \pi_* R$. If $\mathrm{X}(n+1)_*\alpha$ is nilpotent, then $G_k \wedge \alpha^{-1}R \simeq *$ for sufficiently large k.*

as a step towards proving [DHS88, Theorem 1.ii)].

Proof. We want to prove that $\pi_*\left(G_k \wedge \alpha^{-1}R\right) = 0$. Note that

$$\pi_*\left(G_k \wedge \alpha^{-1}R\right) \cong \pi_*\left(\alpha^{-1}\left(G_k \wedge R\right)\right) \cong \alpha^{-1}\pi_*\left(G_k \wedge R\right).$$

The first isomorphism follows from the commutativity of smash products and homotopy colimits. We have the second isomorphism because homotopy groups commute with filtered homotopy colimits. Thus we want to prove

Claim 0.2. *For any $\beta \in \pi_*\left(G_k \wedge R\right)$, there exists $m \in \mathbb{Z}$ such that $\beta\alpha^m = 0$.*

Proof. There exist strongly convergent $\mathrm{X}(n+1)_*$-based Adams–Novikov spectral sequences:

$$\mathrm{Ext}_{\mathrm{X}(n+1)_*\mathrm{X}(n+1)}\left(\mathrm{X}(n+1)_*, \mathrm{X}(n+1)_*R\right) \Rightarrow \pi_*R \tag{0.1}$$

$$\mathrm{Ext}_{\mathrm{X}(n+1)_*\mathrm{X}(n+1)}\left(\mathrm{X}(n+1)_*, \mathrm{X}(n+1)_*(G_k \wedge R)\right) \Rightarrow \pi_*(G_k \wedge R), \tag{0.2}$$

which are compatible with the right action of π_*R on $\pi_*\left(G_k \wedge R\right)$.

Notation 0.3. We use the abbreviation $X \coloneqq X(n+1)$.

Since $\mathrm{X}(n+1)_\alpha$ is nilpotent, we can assume with out loss of generality that $\mathrm{X}(n+1)_*\alpha = 0$. Therefore, α is detected by an element

$$a \in \mathrm{Ext}^{s,t}_{X_*X}\left(X_*, X_*R\right), \text{ for } s > 0.$$

Indeed, denote by Ω^* the cobar resolution of $X(n+1)_*R$, we have a commutative diagram

$$\begin{array}{ccccccc} \pi_*R & \xrightarrow{\mathrm{id}} & \pi_*R \cong \mathbb{S}_*\mathbb{S} \otimes_{\mathbb{S}_*} R & & & & \\ \downarrow & & \downarrow & & & & \\ X_*R & \longrightarrow & \Omega^0 = X_*X \otimes_{X_*} R & \longrightarrow & \Omega^1 & \longrightarrow & \cdots \end{array},$$

which sends α to $1 \otimes_A \alpha$ in Ω^0 through the upper horizontal map.

The element β is detected by an element

$$b \in \mathrm{Ext}^{u,v}_{X_*X}\left(X_*, X_*\left(G_k \wedge R\right)\right).$$

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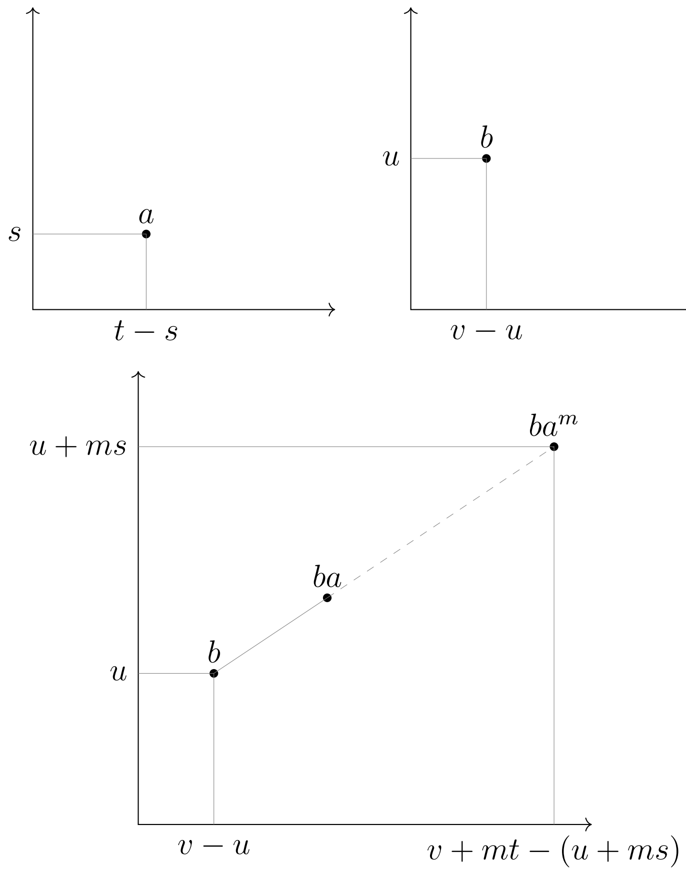
Let us draw a, b, and ba^m in the E^2-pages of the spectral sequences



If we could prove that

i) $ba^m = 0$ for some m, and

ii) the terms in the E^2-page lying above ba^m all vanish,

we would have that $\beta\alpha^m = 0$, because of the strong convergence of the spectral sequences 0.1 and 0.2. The second condition is equivalent to

$$\mathrm{Ext}^{u+ms+j,v+mt+j}_{X_*X}\left(X_*, X_*(G_k \wedge R)\right) = 0, \text{ for all } j \geq 0.$$

These two conditions are satisfied due to the following

Proposition 0.4 ([DHS88, Proposition 2.8]). *Let M be a connective* $\mathrm{X}(n+1)_*\mathrm{X}(n+1)$-*comodule of finite type. Then*

$$\mathrm{Ext}^{s,t}_{\mathrm{X}(n+1)_*\mathrm{X}(n+1)}\left(\mathrm{X}(n+1)_*, \mathrm{X}(n+1)_*G_k \otimes_{\mathrm{X}(n+1)_*} M\right) \tag{0.3}$$

has a vanishing line of slope tending to 0 as k goes to infinity, i.e. there is a constant c and a function $f(k)$ with $\lim_{k\to\infty} f(k) = \infty$ so that the above Ext-*group vanishes whenever $t - s < f(k)s - c$.*

To use this proposition for our proof, we set $M := X_*R$. Note that we have

$$X_*G_k \otimes_{X_*} X_*R \cong X_*\left(G_k \wedge R\right),$$

because X_*G_k is a flat X_*-module. Thus, the Proposition 0.4 can be applied to the spectral sequence 0.2. We choose k such that the slope of the vanishing line is less than $|s(t-s)^{-1}|$. Then there is an m that satisfying $ba^m = 0$ and the temrs in the E^2-page lying above ba^m all vanish. Therefore, we have $\beta\alpha^m = 0$ for sufficiently large m. □

□

Now we would like to present the proof of Proposition 0.4. For this purpose, we need

Proposition 0.5. *Let K be a commutative ring, and $f\colon A\to B$ a map of augmented K-coalgebras. We consider A equipped with a right B-comodule structure induced by f. If A is flat over K and is an extended B-comodule (i.e. $A=C\otimes_K B$), then*

$$\mathrm{Ext}_A\left(K, A\square_B N\right)\cong \mathrm{Ext}_B\left(K,N\right)$$

for any left B-comodule N.

Proof. See the proof of [DHS88, Proposition 2.9]. □

Recall that

Definition 0.6. Let B be an coalgebra over K, A be a right B-comodule and N be a left B-comodule. The *cotensor product* $A\square_B N$ is defined as

$$A\square_B N:=\ker\left(A\otimes_K N\xrightarrow{\psi_A\otimes\, id_N-\mathrm{id}_A\otimes\psi_N} A\otimes_K B\otimes_K N\right),$$

where ψ_A and ψ_N are the coaction maps.

Lemma 0.7. *Let B be a coalgebra over K, S be a right B-comodule, X be a K-module. We have*

i) $S\square_B B\cong S$,

ii) $S\otimes_K X\cong (S\square_B B)\otimes_K X\cong S\square_B\left(B\otimes_K X\right)$, *and*

iii) *for $A=C\otimes_K B$ an extended right B-comodule and L a left B-comodule, we have*

$$A\square_B L=\left(C\otimes_K B\right)\square_B L\cong C\otimes_K\left(B\square_B L\right)\cong C\otimes_K L.$$

Proof of Proposition 0.4. The idea of the proof is to reduce the vanishing line for the Ext-group 0.3 to the vanishing line for

$$\mathrm{Ext}_{\mathbb{F}_p\left[b_n^{p^k}\right]}\left(\mathbb{F}_p,\mathbb{F}_p\right). \tag{0.4}$$

Assume that we made this reduction already and we calculate the Ext-group 0.4 using the normalised cobar complex, *cf.* [MRW77, Note 1.15]. We see that this Ext-group has a vanishing line of slope $\left(2np^k-1\right)^{-1}$, because $b_n^{p^k}$ has dimension $2np^k$. This then completes the proof of the proposition.

Now let us explain how to do the reduction from Ext-group 0.3 to Ext-group 0.4. First, let us recall that for a split Hopf algebroid $\left(A, A\tilde{\otimes}S\right)$ over a commutative ring K, we have

$$\mathrm{Ext}_{A\tilde{\otimes}S}\left(A,M\right)\cong\mathrm{Ext}_S\left(K,M\right).$$

Thus, using this fact together with [DHS88, Proposition 2.4] and [DHS88, Proposition 2.5], we have that the Ext-group 0.3 is isomorphic to

$$\mathrm{Ext}_{\mathbb{Z}_{(p)}[b_1,\ldots,b_n]}\left(\mathbb{Z}_{(p)},\mathbb{Z}_{(p)}[b_1,\ldots,b_{n-1}]\{1,b_n,\ldots,b_n^{p^k-1}\}\otimes_{\mathbb{Z}_{(p)}}M\right). \tag{0.5}$$

Now we would like to simplify the terms in the above Ext-group. Let $\mathbb{Z}_{(p)}[b_n]$ be a Hopf algebra with b_n primitive. Using Lemma 0.7, we have

$$\begin{aligned}
&\mathbb{Z}_{(p)}[b_1,\ldots,b_{n-1}]\{1,b_n,\ldots,b_n^{p^k-1}\}\\
\cong&\mathbb{Z}_{(p)}[b_1,\ldots,b_n-1]\otimes_{\mathbb{Z}_{(p)}}\left(\mathbb{Z}_{(p)}[b_n]\square_{\mathbb{Z}_{(p)}[b_n]}\mathbb{Z}_{(p)}\{1,b_n,\ldots,b_n^{p^k-1}\}\right)\\
\cong&\left(\mathbb{Z}_{(p)}[b_1,\ldots,b_{n-1}]\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_{(p)}[b_n]\right)\square_{\mathbb{Z}_{(p)}[b_n]}\mathbb{Z}_{(p)}\{1,b_n,\ldots,b_n^{p^k-1}\}\\
\cong&\mathbb{Z}_{(p)}[b_1,\ldots,b_n]\square_{\mathbb{Z}_{(p)}[b_n]}\mathbb{Z}_{(p)}\{1,b_n,\ldots,b_n^{p^k-1}\}.
\end{aligned}$$

Thus we have

$$\begin{aligned}&\mathbb{Z}_{(p)}[b_1,\dots,b_n-1]\{1,b_n,\dots,b_n^{p^k-1}\}\otimes_{\mathbb{Z}_{(p)}} M\\ \cong&\mathbb{Z}_{(p)}[b_1,\dots,b_n]\square_{\mathbb{Z}_{(p)}[b_n]}\mathbb{Z}_{(p)}\{1,b_n,\dots,b_n^{p^k-1}\}\otimes_{\mathbb{Z}_{(p)}} M\\ \cong&\mathbb{Z}_{(p)}[b_1,\dots,b_n]\square_{\mathbb{Z}_{(p)}[b_n]}\left(\mathbb{Z}_{(p)}\{1,b_n,\dots,b_n^{p^k-1}\}\otimes_{\mathbb{Z}_{(p)}} M\right).\end{aligned}$$

Therefore, apply Proposition 0.5 to the Ext-group 0.5, we have an isomorphic Ext-group

$$\mathrm{Ext}_{\mathbb{Z}_{(p)}[b_n]}\left(\mathbb{Z}_{(p)},\mathbb{Z}_{(p)}\left\{1,b_n,\dots,b_n^{p^k-1}\right\}\otimes_{\mathbb{Z}_{(p)}} M\right). \tag{0.6}$$

There is a May spectral sequence approximating the Ext-group 0.6, which has E_2-page

$$\mathrm{Ext}_{\mathbb{F}_p[b_n]}\left(\mathbb{F}_p,\mathbb{F}_p\left\{1,b_n,\dots,b_n^{p^k-1}\right\}\otimes_{\mathbb{F}_p} E_0M\right).$$

This spectral sequence is constructed by using the p-adic filtration of the cobar resolution $\Omega^*\left(\mathbb{Z}_{(p)},\mathbb{Z}_{(p)}\left\{1,b_n,\dots,b_n^{p^k-1}\right\}\otimes_{\mathbb{Z}_{(p)}} M\right)$, *cf.* [Mil81, Chapter 8]. Here, E_0M denotes the bigraded object formed from the successive quotients of the p-adic filtration. Applying the convergence results [EM62, Corollary 6.3] together with [DHS88, Lemma 2.6], it suffices to have a vanishing line with desired slope for

$$\mathrm{Ext}_{\mathbb{F}_p[b_n]}\left(\mathbb{F}_p,\mathbb{F}_p\left\{1,b_n,\dots,b_n^{p^k-1}\right\}\right).$$

As coalgebras, we have

$$\begin{aligned}\mathbb{F}_p[b_n]&\cong\otimes_{j\geq 0}\,\mathbb{F}_p\left[b_n^{p^j}\right]\Big/\left(b_n^{p^{j+1}}\right)\\ \mathbb{F}_p\left\{1,b_n,\dots,b_n^{p^k-1}\right\}&\cong\otimes_{j<k}\,\mathbb{F}_p\left[b_n^{p^j}\right]\Big/\left(b_n^{p^{j+1}}\right).\end{aligned}$$

Apply Proposition 0.5, we have

$$\mathrm{Ext}_{\mathbb{F}_p[b_n]}\left(\mathbb{F}_p,\mathbb{F}_p\left\{1,b_n,\dots,b_n^{p^k-1}\right\}\right)\cong\mathrm{Ext}_{\otimes_{j\geq k}\mathbb{F}_p\left[b_n^{p^j}\right]/\left(b_n^{p^{j+1}}\right)}\left(\mathbb{F}_p,\mathbb{F}_p\right)\cong\mathrm{Ext}_{\mathbb{F}_p\left[b_n^{p^k}\right]}\left(\mathbb{F}_p,\mathbb{F}_p\right).$$

Therefore, we reduce the original proposition to the vanishing line for $\mathrm{Ext}_{\mathbb{F}_p\left[b_n^{p^k}\right]}(\mathbb{F}_p,\mathbb{F}_p)$. □

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