ORTHOGONAL SPECTRA

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This is my talk at European Autumn School in Topology 2019. I mainly used [Sch18] and [Sch19] for preparing the talk.

1. Motivation for spectra

On the one hand, there are stable phenomena in the category of topological spaces, for example:

Theorem 1.1 (Freudenthal suspension theorem). *Let X be a n-connected pointed topological spaces, then we have*

$$\pi_{n+k}(X) \xrightarrow{\cong} \pi_{n+k+1}(\Sigma X)$$

for $n \geq k$.

Thus, we may want "stable objects" to make the suspension functor to a real equivalence of categories.

On the other hand, let us consider generalised cohomology theories. Denote by $\widetilde{\mathrm{H}}(-;A)$ the reduced singular cohomology theory with coefficient abelian group A. We know that for all $n \in \mathbb{N}$, we have $\widetilde{\mathrm{H}}^n(X;A) \cong [X, K(A,n)]_{\mathrm{pt}}$ for all pointed spaces X. Suspension axiom tells us that for all n, we have

$$\widetilde{\mathrm{H}}^n(X;A) \cong \widetilde{\mathrm{H}}^{n+1}(\Sigma X;A).$$

Thus we have

$$[X, K(A,n)]_{\mathrm{pt}} \cong [\Sigma X, K(A,n+1)]_{\mathrm{pt}} \overset{\Sigma - \Omega}{\cong} [X, \Omega K(A,n+1)]_{\mathrm{pt}}.$$

Applying Yoneda lemma, we obtain an homotopy equivalence $K(A,n) \xrightarrow{\simeq} \Omega K(A,n+1)$. Therefore, we can interpret reduced singular cohomology theory as a sequence of pointed spaces $\{K(A,n)\}_{n\in\mathbb{N}}$ with "structure maps" $\{K(A,n) \xrightarrow{\simeq} \Omega K(A,n+1)\}_{n\in\mathbb{N}}$.

Definition 1.2. A *prespectrum* E consists of

i) A sequence $\{E_n\}_{n\geq 0}$ of pointed spaces,

ii) Structure maps $\sigma_n \colon \Sigma E_n \to E_{n+1}$ for all $n \in \mathbb{N}$.

Remark 1.3. Prespectra turns out to be one of the objects that can both explain the stable phenomena in spaces, and represent (generalised) cohomology theories:

i) Let X be a topological space. We can define the *suspension spectrum* $\Sigma^\infty X$ with $(\Sigma^\infty X)_n \coloneqq \Sigma^n X$ and the structure maps are the identity maps. The homotopy group of $\Sigma^\infty X$ is the "stabilisation" of the homotopy group of X under the suspension functor, *cf.* Definition 4.1.

   Notation 1.4. Denote by $\mathbb{S}$ the suspension spectrum $\Sigma^\infty \mathrm{S}^0$.

ii) The Brown representability theorem shows that generalised cohomology theories relates closely to prespectra (next talk).

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Problem 1.5. It is hard to define symmetric monoidal structure (smash products) on the category of prespectra, extending the smash product construction on spaces.

There are several motivations of having a symmetric monoidal structure. For example, we could then define algebra objects and do algebras in the category of (pre)spectra. Also note that, for singular *homology* theory, we have

$$\mathrm{H}_k(X;A) \cong \pi_{n+k}(X \wedge K(A,n)),$$

for n large enough. Therefore, if we could define smash product, we may also be able to define homology theories using spectra.

2. Orthogonal Spectra

In order to be able to define smash product, we need to introduce one of the alternative models for spectra: orthogonal spetra.

Definition 2.1. An orthogonal spectrum is a prespectrum $E = \{E_n\}_{n\in\mathbb{N}}$ with the following data: there is a base point preserving continuous left action of the orthogonal group $\mathrm{O}(n)$ on E_n for all $n \in \mathbb{N}$, such that for all n, m in $\mathbb{N}$, the *iterated structure map* σ^m

$$\begin{array}{ccccccc} E_n \wedge \mathrm{S}^m & & & \xrightarrow{\quad\sigma^m\quad} & & & E_{n+m} \\ & \searrow & & & & \nearrow & \\ & & E_{n+1} \wedge \mathrm{S}^{m-1} & \longrightarrow \cdots \longrightarrow & E_{n+m-1} \wedge \mathrm{S}^1 & & \end{array}$$

is $\mathrm{O}(n) \times \mathrm{O}(m)$ equivariant.

Definition 2.2. A *morphism* $f\colon E \to E'$ of orthogonal spectra consists of a collection

$$\{f_n\colon E_n \xrightarrow{\mathrm{O}(n)} E'_n \in \mathbf{Top}_* \mid f_n \text{ is } \mathrm{O}(n) \text{ equivariant } \}_{n\in\mathbb{N}}$$

of maps of pointed spaces such that the following diagram commutes for all $n \in \mathbb{N}$

$$\begin{array}{ccc} E_n \wedge \mathrm{S}^1 & \xrightarrow{\sigma_n^E} & E_{n+1} \\ \big\downarrow{\scriptstyle f_n \wedge \mathrm{id}} & & \big\downarrow{\scriptstyle f_{n+1}} \\ E'_n \wedge \mathrm{S}^1 & \xrightarrow{\sigma_n^{E'}} & E'_{n+1} \end{array}.$$

Notation 2.3. Denote the category of orthogonal spectra by $\mathbf{Sp}^{\mathrm{O}}$.

Example 2.4 (Suspension spectra)**.** Let X be a topological space. We have seen the definition of $\Sigma^\infty X$ as a prespectrum. In order to make $\Sigma^\infty X$ to a orthogonal spectrum, we set $(\Sigma^\infty X)_n := X \wedge \mathrm{S}^n$. This way, $(\Sigma^\infty X)_n$ has an left $\mathrm{O}(n)$ action via the canonical action on $\mathrm{S}^n \approx \mathbb{R}^+$, where $(-)^+$ denotes the one point compactification.

Example 2.5 (Eilenberg–Maclane Spectra)**.** We can define the Eilenberg–Maclane spectra HA for an abelian group A as a prespectrum via setting $(\mathrm{HA})_n := K(A,n)$. However, how can we give $K(A,n)$ an $\mathrm{O}(n)$-action?

Definition 2.6. Define the space $A[\mathrm{S}^n]$ of A-*linearisation* of (S^n, s_0) as

$$A[\mathrm{S}^n] \overset{\mathbf{Set}}{:=} \frac{\{\sum_{i=1}^m a_i v_i \mid a_i \in A, v_i \in \mathrm{S}^n, m < \infty\}}{(a_1 v + a_2 v = (a_1 + a_2)v, \text{ and } a s_0 = 0, \forall a \in A)}$$

with the quotient topology induced from $\coprod_{m\geq 0} A^m \times (\mathrm{S}^n)^m$ via the surjection

$$\coprod_{m\geq 0} A^m \times (\mathrm{S}^n)^m \twoheadrightarrow A[\mathrm{S}^n],\ (a_1,\ldots,a_m,v_1,\ldots,v_m) \mapsto \sum_{i=1}^m a_i v_i.$$

Proposition 2.7. *We have $A[\mathrm{S}^n] \simeq K(A,n)$.*

Now we can define the Eilenber–Maclane spectra HA as an orthogonal spectrum by setting $(\mathrm{HA})_n \coloneqq A[\mathrm{S}^n]$ with structure map

$$\sigma_n \colon A[\mathrm{S}^n] \wedge \mathrm{S}^1 \to A[\mathrm{S}^{n+1}],\ \left(\sum_{i=1}^m a_i v_i \wedge w\right) \mapsto \sum_{i=1}^m a_i(v_i \wedge w).$$

The action of $\mathrm{O}(n)$ on $(\mathrm{HA})_n$ is given again by the canonical action of S^n.

Example 2.8 (Unoriented bordism MO)**.** The unoriented bordism spectrum MO corresponds[1] to the unoriented bordism homology theory with

$$\mathrm{MO}_m(X) \cong \{f\colon M \to X \mid M \text{ is a } m\text{-dimensional manifold}\}/\text{ bordisms},$$

the *n-dimensional unoriented bordism group of X*. By manifolds we meant smooth and closed manifolds.

As a orthogonal spectrum, MO is define via

$$\mathrm{MO}_n \coloneqq \mathrm{EO}(n)_+ \wedge_{\mathrm{O}(n)} \mathrm{S}^n,$$

the Thom space of the tautological vector bundle $\gamma^n_{\mathbb{R}}\colon \mathrm{EO}(n) \times_{\mathrm{O}(n)} \mathbb{R}^n \to \mathrm{BO}(n)$. Recall that $\mathrm{O}(n)$ acts one the left and on the right of $\mathrm{EO}(n)$ and S^n. Thus the left action of $\mathrm{O}(n)$ on $(\mathrm{MO})_n$ is given by the remaining left action of $\mathrm{O}(n)$ on $\mathrm{EO}(n)$.

Now let us consider the structure map: we have the following pullback of vector bundles

$$\begin{array}{ccc} \mathrm{EO}(n)_+ \times_{\mathrm{O}(n)} \mathbb{R}^n \times \mathbb{R} & \xrightarrow{i^*} & \mathrm{EO}(n+1)_+ \times_{\mathrm{O}(n+1)} \mathbb{R}^{n+1} \\ \downarrow & & \downarrow{\scriptstyle \gamma^{n+1}_{\mathbb{R}}} \\ \mathrm{BO}(n) & \xrightarrow{i} & \mathrm{BO}(n+1) \end{array},$$

where the map i is induced by canonical inclusion of n-dimensional vector subspaces to $n+1$-dimensional vector subspace. Then we define the induced map

$$\sigma_n \coloneqq \mathrm{Th}(i^*)\colon \mathrm{EO}(n)_+ \wedge_{\mathrm{O}(n)} \mathrm{S}^n \wedge \mathrm{S}^1 \to \mathrm{EO}(n+1)_+ \wedge_{\mathrm{O}(n+1)} \mathrm{S}^{n+1}$$

on the Thom spaces to be the structure map for all $n \in \mathbb{N}$.

Example 2.9 (Complex cobordism MU)**.** The complex cobordism spectrum MO corresponds to the complex bordism homology theory with

$$\mathrm{MU}_m(X) \cong \{f\colon M \to X \mid M \text{ is a } m\text{-dimensional stably complex manifold}\}/\text{ bordisms}.$$

Recall that a manifold is stably complex if the direct sum of its tangent bundle and some trivial vector bundles is isomorphic to a complex vector bundle.

Analogue to MO, our first attempt is to define the sequence of pointed spaces

$$\overline{\mathrm{MU}}_n \coloneqq \mathrm{EU}(n)_+ \wedge_{\mathrm{U}(n)} \mathrm{S}^{\mathbb{C}^n},$$

which is the Thom space of the tautological bundle $\gamma^n_{\mathbb{C}}\colon \mathrm{EU}(n) \times_{\mathrm{U}(n)} \mathbb{C}^n \to \mathrm{BU}(n)$. Here $\mathrm{S}^{\mathbb{C}^n} \coloneqq (\mathbb{C}^n)^+ \approx \mathrm{S}^{2n}$.

[1]using the Brown representability theorem

Let us consider the $\mathrm{O}(n)$-action now: Write $\mathbb{C}^n \cong \mathbb{R}^n \oplus i\mathbb{R}^n$. Then the $\mathrm{O}(n)$ action on $\mathbb{C}^n$ is given by the canonical action on $\mathbb{R}^n$ componentwise. Furthermore, this is a unitary action. In this way, we can consider $\mathrm{O}(n)$ as a subgroup of $\mathrm{U}(n)$, and thus $\mathrm{O}(n)$ acts on the left and right on $\mathrm{EU}(n)$ by restriction. Therefore, $\mathrm{O}(n)$ acts on the left on $\overline{\mathrm{MU}}_n$ via the remaining left action on $\mathrm{EU}(n)$.

How about the structure map? We have a canonical inclusion $j\colon \mathrm{BU}(n) \to \mathrm{BU}(n+1)$, analog to MO. Therefore we can obtain an induced map

$$\mathrm{Th}(j^*)\colon \overline{\mathrm{MU}}_n \wedge \mathrm{S}^2 = \mathrm{EU}(n)_+ \wedge_{\mathrm{U}(n)} \wedge \mathrm{S}^{\mathbb{C}^n} \wedge \mathrm{S}^{\mathbb{C}} \to \mathrm{EU}(n+1)_+ \wedge_{\mathrm{U}(n+1)} \mathrm{S}^{\mathbb{C}^{n+1}} = \overline{\mathrm{MU}}_{n+1}.$$

on the Thom spaces. In other words, we don't have a canonical map $\overline{\mathrm{MU}}_n \wedge \mathrm{S}^1 \to \overline{\mathrm{MU}}_{n+1}$. To fix this problem, we define the sequences of pointed spaces

$$\mathrm{MU}_n \coloneqq \mathrm{Map}\left(\mathrm{S}^{i\mathbb{R}}, \overline{\mathrm{MU}}_n\right), \text{ for } n \in \mathbb{N}$$

The left $\mathrm{O}(n)$ action is give by conjugation, *i.e.* $Mf(v) = M(f(M^{-1}v))$ for all $M \in \mathrm{O}(n)$ and all $v \in \mathrm{S}^{i\mathbb{R}}$. The structure map is

$$\begin{aligned}
\sigma_n\colon \mathrm{MU}_n \wedge \mathrm{S}^{\mathbb{R}} &= \mathrm{Map}\left(\mathrm{S}^{i\mathbb{R}}, \overline{\mathrm{MU}}_n\right) \wedge \mathrm{S}^{\mathbb{R}} \\
&\xrightarrow{\mathrm{ev}} \mathrm{Map}\left(\mathrm{S}^{i\mathbb{R}^n}, \overline{\mathrm{MU}}_n \wedge \mathrm{S}^{\mathbb{R}}\right) \\
&\xrightarrow{(*)} \mathrm{Map}\left(\mathrm{S}^{i\mathbb{R}^n}, \mathrm{Map}\left(\mathrm{S}^{i\mathbb{R}^n}, \overline{\mathrm{MU}}_{n+1}\right)\right) \\
&\cong \mathrm{Map}\left(\mathrm{S}^{i\mathbb{R}^n}, \overline{\mathrm{MU}}_{n+1}\right) \\
&= \mathrm{MU}_{n+1},
\end{aligned}$$

where the map $(*)$ is induced by the adjunction of the map (replacing $\mathbb{C}$ by $\mathbb{R}^n \oplus i\mathbb{R}^n$)

$$\mathrm{Th}(j^*)\colon \overline{\mathrm{MU}}_n \wedge \mathrm{S}^{\mathbb{R}} \wedge \mathrm{S}^{i\mathbb{R}^n} \to \overline{\mathrm{MU}}_{n+1}.$$

We can check that the $\mathrm{MU} \coloneqq (\mathrm{MU}_n, \sigma_n)_{n\in\mathbb{N}}$ with the described $\mathrm{O}(n)$ action is an orthogonal spectrum.

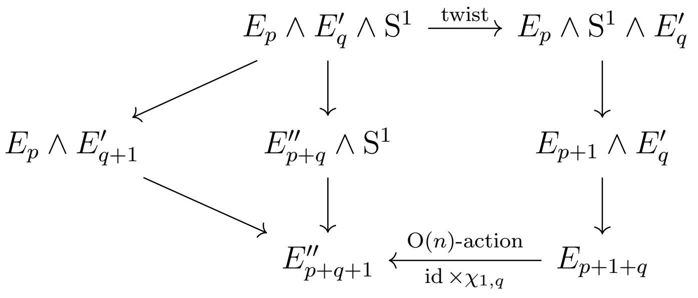
3. Symmetric monoidal structure

In section, we are going to define the smash product of two orthogonal spectra. Intuitively, you shall think of smash product of spectra as analogue to vector spaces with tensor product. Thus we are going to first define bimorphisms in $\mathbf{Sp}^{\mathrm{O}}$, which is analogue to a bilinear map of vector spaces.

Definition 3.1. A *bimorphism* $b\colon (E, E') \to \mathrm{E}''$ with orthogonal spectra E, E' and E'' is a collection

$$\{b_{p,q}\colon E_p \wedge E'_q \to E''_{p+q} \in \mathbf{Top}_* \mid b_{p,q} \text{ is } \mathrm{O}(p) \times \mathrm{O}(q) \text{ equivariant } \}_{p,q\in\mathbb{N}}$$

of maps such that the following diagram commutes



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where $\chi_{1,q}\colon \mathbb{R}^{1+q} \to \mathbb{R}^{q+1}$, $(v_1, v_2, \dots, v_{q+1}) \mapsto (v_2, \dots, v_{q+1}, v_1)$.

Definition 3.2. A *smash product* for $E, E' \in \mathbf{Sp}^{\mathrm{O}}$ is a pair $(E \wedge E', i)$ with $E \wedge E' \in \mathbf{Sp}^{\mathrm{O}}$ and a bimorphism $i\colon (E, E') \to E \wedge E'$ such that the map induced by i:

$$\mathbf{Sp}^{\mathrm{O}}\left(E \wedge E', E''\right) \xrightarrow{i_*} \mathrm{Bimor}\left((E, E'), E''\right)$$

is an bijection for any $E'' \in \mathbf{Sp}^{\mathrm{O}}$.

Proposition 3.3. *For any two $E, E' \in \mathbf{Sp}^{\mathrm{O}}$, the smash product $E \wedge E'$ exists.*

Proposition 3.4. *The smash product is symmetric monoidal, i.e. it satisfies*

i) *(associativity)* $(E \wedge E') \wedge E'' \xrightarrow{\cong} E \wedge (E' \wedge E'')$,

ii) *(commutativity)* $E \wedge E' \xrightarrow{\cong} E' \wedge E$,

iii) *(unitarity)* $\mathbb{S} \wedge E \cong E \cong E \wedge \mathbb{S}$,

for any $E, E', E'' \in \mathbf{Sp}^{\mathrm{O}}$.

Proposition 3.5. *The category $\left(\mathbf{Sp}^{\mathrm{O}}, \wedge, \mathbb{S}\right)$ is a closed*[2] *symmetric monoidal category.*

Definition 3.6. A *ring spectrum* is a monoid in $\left(\mathbf{Sp}^{\mathrm{O}}, \wedge, \mathbb{S}\right)$.

Remark 3.7. More concretely, a ring spectrum $R \in \mathbf{Sp}^{\mathrm{O}}$ is an orthogonal spectrum R together with a morphism $R \wedge R \to R$ of orthogonal spectra satisfying certain conditions (associativity etc.) By Definition 3.2, we can equivalently consider the map $R \wedge R \to R$ as a bimorphism $(R, R) \to R$.

We are going to see that the orthogonal spectra introduced in Example 2.4, 2.5, 2.8, and 2.9 are all[3] ring spectra. To show this, we are only going to write down the bimorphisms and leave the reader to check the rest.

Example 3.8 (Eilenberg–Maclane spectra)**.** Let A be a ring. Then the Eilenberg–Maclane spectrum HA is a ring spectrum, because, for each $m, n \in NN$, we can define the bimorphism

$$\mathrm{HA}_m \wedge \mathrm{HA}_n \to \mathrm{H(A \otimes A)}_{m+n} \to \mathrm{HA}_{m+n}, \ \forall\, m, n \in \mathbb{N}$$

$$\left(\sum_i a_i v_i\right) \wedge \left(\sum_j b_j w_j\right) \mapsto \left(\sum_{i,\ j} a_i \otimes b_j v_i \wedge w_j\right) \mapsto \sum_{i,\ j} a_i b_j v_i \wedge w_j.$$

If A is commutative, then HA is a *commutative ring spectrum*, *i.e.* a commutative monoid in $\left(\mathbf{Sp}^{\mathrm{O}}, \wedge, \mathbb{S}\right)$.

Example 3.9 (MO)**.** To define the monoid structure on MO, note that there is a map on the tautological bundle

$$\gamma_{\mathbb{R}}^m \times \gamma_{\mathbb{R}}^n \to \gamma_{\mathbb{R}}^{m+n}, \ \forall\, m, n \in \mathbb{N}$$

induced by sending a m-dimensional vector subspace and a n-dimensional vector subspace to the direct sum of these two. The induced map on the Thom spaces gives us a bimorphism

$$\mathrm{MO}_m \wedge \mathrm{MO}_n \to \mathrm{MO}_{m+n}, \ \forall\, m, n \in \mathbb{N}.$$

Example 3.10 (MU)**.** Similar as MO, we can obtain maps (of pointed spaces)

$$\overline{\mathrm{MU}}_m \wedge \overline{\mathrm{MU}}_n \to \overline{\mathrm{MU}}_{m+n}, \ \forall\, m, n \in \mathbb{N}$$

[2]There exists "mapping spectra" $\mathrm{Map}(-,-)$ with the property that $\mathrm{Map}\left(E \wedge E', E''\right) \cong \mathrm{Map}\left(E, \mathrm{Map}\left(E', E''\right)\right)$.

[3]For the suspension spectra, we need to require that A is a ring.

the induced map on the Thom space of the "direct sum map" of the tautological bundles. Thus, we obtain bimorphism

$$\begin{aligned}\mathrm{MU}_m \wedge \mathrm{MU}_n &= \Omega^m\, \overline{\mathrm{MU}}_m \wedge \Omega^n\, \overline{\mathrm{MU}}_n \\ &\xrightarrow{\wedge} \Omega^{m+n}\, \overline{\mathrm{MU}}_m \wedge \overline{\mathrm{MU}}_n \\ &\to \Omega^{m+n}\, \overline{\mathrm{MU}}_{m+n} \\ &= \mathrm{MU}_{m+n},\ \forall\, m, n \in \mathbb{N}.\end{aligned}$$

4. Stable equivalence

Definition 4.1. The *k-th homotopy group* of $E \in \mathbf{Sp}^{\mathrm{O}}$ is $\pi_k(E) \coloneqq \operatorname{colim}_{n\to\infty} \pi_{k+n} E_n$.

Definition 4.2. The homotopy category $\mathbf{Ho}\,\mathbf{Sp}^{\mathrm{O}}$ ($\simeq$ **SHC**[4]) is defined via formally inverting the π_*-isomorphisms in $\mathbf{Sp}^{\mathrm{O}}$. In other words, every functor $F\colon \mathbf{Sp}^{\mathrm{O}} \to \mathcal{C}$ factors through $\mathbf{Ho}\,\mathbf{Sp}^{\mathrm{O}}$ uniquely if and only if F sends π_*-isomorphisms to isomorphisms in $\mathcal{C}$.

Remark 4.3. There are a lot of models for a category of spectra, whose underlying homotopy category are all equivalent. In different models we have different conveniences. In the category of prespectra, it is (most of the time) easy to define a spectra, while in the category of orthogonal spectra, or the category of symmetric spectra, *cf.* [Sch07], we can define symmetric monoidal structures. A good reference for introduction on the stable homotopy category and comparison between different approaches is [Mal14].

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[4]Stable homotopy category.