COMPLEX ORIENTED COHOMOLOGY THEORY AND MU

YUQING SHI

These are the notes for my talk at the Bayerische Klein Arbeitsgemeinschaft titled "Towards Chromatic Homotopy Theory" on 03 March 2019 in Bayreuth.

In this talk we will see some geometric and topological aspects of the formal group laws and the Lazard ring L. It can be summarised into the following theorem of Quillen.

Theorem (Quillen)**.** *There is an isomorphism $L \to MU_*$ of graded rings.*

Here MU_* is the coefficient ring of complex cobordism. The plan of this talk is to motivate and introduce the relevant notions in topology and sketch the rational version of Quillen's theorem. References for this talk are [Ada74], [Lur10] and [Mal14].

1. Complex oriented cohomology theory (COCT)

Let us recall the definitions of the additive and the multiplicative formal group laws. They can be seen as the formulas of the first Chern class of the tensor product of two complex line bundles, where the Chern class takes values in ordinary cohomology and complex K-theory respectively. More precisely, let us consider the multiplication μ on $\mathbb{C}\mathrm{P}^\infty$,

$$\mu\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty,$$

which classifies the tensor product of two complex line bundles. This map induces a map on ordinary cohomology

$$\mu^*\colon \mathbb{Z}[[t]] \cong \mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty) \to \mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}[[x, y]]$$

where $[[\bullet]]$ denotes the formal power series. The additive formal group law is the image of t under the map μ^*. If we replace the ordinary cohomology by complex K-theory in the map μ^*, we obtain the multiplicative formal group law.

Question 1.1. Can we use this procedure to obtain other formal group laws from any other generalised cohomology theory? Or in other words, in which generalised cohomology theory can we define Chern classes and how the corresponding formula of the Chern class of tensor products of two complex line bundles looks?

We will see that the "complex oriented cohomology theories" are the ones that we are looking for.

Definition 1.2. A multiplicative cohomology theory E is *complex orientable* if the map $i^*\colon \widetilde{E}^2(\mathbb{C}\mathrm{P}^\infty) \to \widetilde{E}^2(\mathrm{S}^2)$ induced by the canonical inclusion $i\colon \mathrm{S}^2 \approx \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$ is surjective.

Remark 1.3. By the suspension axiom we have that $\widetilde{E}^2(\mathrm{S}^2) \cong \widetilde{E}^0(\mathrm{S}^0) \cong E^0(\mathrm{pt})$. In $E^0(\mathrm{pt})$ we have the unit element 1 of the multiplication of E. Furthermore, the map i^* is a map of $E^0(\mathrm{pt})$-modules. Thus E is complex orientable if and only if there exists a cohomology class $t \in \widetilde{E}^2(\mathbb{C}\mathrm{P}^\infty)$ such that $i^*(t) = 1$.

Definition 1.4. A choice of such a cohomology class t is called a *complex orientation* of E. We say the pair (E, t) is a *complex oriented cohomology theory*.

Date: 03.03.2019.
y.shi@uu.nl.

Example 1.5.

i) Singular cohomology with the universal first Chern class is complex oriented. The multiplicative structure is given by the cup product.
ii) There exists a "universal" complex oriented cohomology theory (MU, t_{MU}). Given any complex oriented cohomology theory (E, t), there exists a unique ("up to homotopy") "ring" map $(MU, t_{MU}) \to (E, t)$.

Given a complex oriented cohomology theory, we can do some nice computations.

Proposition 1.6. *Let (E, t) be a complex oriented cohomology theory. Denote $E_* := E_*(pt)$. We have:*

i) *a graded ring isomorphism $E^*(\mathbb{C}\mathrm{P}^\infty) \cong E_*[[t]]$, and*
ii) *a graded ring isomorphism $E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong E_*[[x, y]]$, where $x = \mathrm{pr}_1^*(t)$ and $y = \mathrm{pr}_2^*(t)$, and pr_1 and pr_2 are the canonical projections from $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$ to $\mathbb{C}\mathrm{P}^\infty$.*

Remark 1.7. The above isomorphisms depend on E and the choice of orientation t.

Thus the multiplication μ on $\mathbb{C}\mathrm{P}^\infty$ induces a map

$$\mu^* \colon E_*[[t]] \cong E^*(\mathbb{C}\mathrm{P}^\infty) \to E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong E_*[[x, y]].$$

Remark 1.8. Denote by $f_{E,t}(x, y)$ the image of t under μ^*.

i) The formal power series $f_{E,t}(x, y)$ is a formal group law over the ring E_*, because the multiplication μ is unital, associative and commutative.
ii) The Landweber exact functor theorem tells us when a formal group law comes from a complex oriented cohomology theory in this way.

2. Universal complex oriented cohomology theory (MU, t_{MU})

First, let us introduce an alternative way to think about cohomology theories.

Theorem 2.1 (Brown representability)**.** *Let h be a generalised cohomology theory. Then for every $n \in \mathbb{Z}$, there exists a pointed space E_n such that $\widetilde{h}^n(X) \cong [X, E_n]_{\mathrm{pt}}$. Here $[\bullet]$ denotes the homotopy classes of maps.*

By suspension axiom we have $\widetilde{h}^n(X) \cong \widetilde{h}^{n+1}(\Sigma X)$, where Σ denotes the reduced suspension. Interpreting this isomorphism by Brown representability, we obtain

$$[X, E_n]_{\mathrm{pt}} \cong [\Sigma X, E_{n+1}]_{\mathrm{pt}} \cong [X, \Omega E_{n+1}]_{\mathrm{pt}},$$

where the second isomorphism is given by the $\Sigma - \Omega$ adjunction. Thus we obtain a homotopy equivalence $E_n \xrightarrow{\simeq} \Omega E_{n+1}$, which corresponds again by adjunction to a map $\Sigma E_n \to E_{n+1}$ (this map does not have to be a homotopy equivalence).

Remark 2.2. Thus we can think of a generalised cohomology theory as a sequence of pointed spaces together with some maps. This is essentially the definition of a prespectrum.

Definition 2.3. A *prespectrum* $E = \{E_n\}_{n \in \mathbb{Z}}$ is a sequence of pointed spaces together with structure maps $\Sigma E_n \to E_{n+1}$. A *morphism* f between two prespectra E and E' consists of maps $f_n \colon E_n \to E'_n$ such that they are compatible with the structure maps. More concretely, the following diagram of pointed spaces commutes.

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\Sigma f_n} & \Sigma E'_n \\ \downarrow & & \downarrow \\ E_{n+1} & \xrightarrow[f_{n+1}]{} & E'_{n+1}. \end{array}$$

This gives us the category of prespectra, denoted by **PrSp**.

Remark 2.4. We don't require that the adjoint of the structure map to be a homotopy equivalence. If it is, we call the object an Ω-prespectrum.

Definition 2.5. Let E be a prespectrum.

i) Define the *homotopy group* $\pi_n E \coloneqq \operatorname{colim}_{i\in\mathbb{Z}} \pi_{k+i} E_n$, for each $n \in \mathbb{Z}$.
ii) Define the *homotopy category* **HoSp** *of spectra* as the category **PrSp** "localised at π_*-isomorphisms", where π_*-isomorphisms are morphisms that induce isomorphisms between all homotopy groups.

Remark 2.6. Our definition of **HoSp** is not precise enough. However, this is how you should think about it: the objects are prespectra and the morphisms are "homotopy classes of morphisms between prespectra", denoted by $[\bullet, \bullet]$. It actually takes some work to define this category rigorously, see [Mal14] for a detailed explanation.

Remark 2.7. From now on we will work in the **HoSp** and most of the time we will only consider its abstract definitions and properties instead of analysing the objects and morphisms concretely.

Proposition 2.8.

i) *There is a* suspension spectrum functor $\Sigma^\infty\colon \mathbf{HoTop}_* \to \mathbf{HoSp}, X \mapsto \Sigma^\infty X$, *defined by* $(\Sigma^\infty X)_n \coloneqq \Sigma^n X$. *Define the* sphere spectrum $\mathbb{S}$ *as* $\Sigma^\infty \mathrm{S}^0$.
ii) *There is a* shift functor $\Sigma\colon$ **HoSp** $\to$ **HoSp**, *defined by* $(\Sigma E)_n \coloneqq E_{n+1}$. *This is an equivalence of categories.*
iii) *There is a smash product* $\wedge$ *on* **HoSp** *such that* $(\mathbf{HoSp}, \wedge, \mathbb{S})$ *is a symmetric monoidal category.*

Definition 2.9. A *ring spectrum* is a monoid in $(\mathbf{HoSp}, \wedge, \mathbb{S})$.

Example 2.10. Let $\gamma_n\colon EU(n) \to BU(n)$ be the universal principal $U(n)$-bundle, denote by $\mathrm{Th}(\gamma_n)$ the Thom space of the associated universal complex vector bundle of rank n. For $n \geq 0$, define the spectrum

$$MU(n) \coloneqq \Sigma^{\infty-2n} \mathrm{Th}(\gamma) \coloneqq \Sigma^{-2n} \Sigma^\infty \mathrm{Th}(\gamma_n).$$

For every $n \geq 1$, we have a map $\iota_n\colon MU(n-1) \to MU(n)$ that is induced by the map $BU(n-1) \to BU(n)$ which classifies the Whitney sum with a trivial complex line bundle. Take the homotopy colimit over these maps:

$$MU(0) \to MU(1) \to \cdots \mathrm{MU}(n) \to \cdots,$$

from which we obtain the spectrum

$$MU \coloneqq \operatorname{hocolim}_{n\geq 0} MU(n).$$

Furthermore, MU is a ring spectrum. Indeed, for $a \geq 0$ and $b \geq 0$, the map $BU(a) \times BU(b) \to BU(a+b)$, which classifies the Whitney sum of two complex vector bundles of rank a and b, induces a map

$$MU(a) \wedge MU(b) \to MU(a+b).$$

Upon taking the colimit, we obtain the multiplication $MU \wedge MU \to MU$.

Definition 2.11. Given a spectrum E and a space X, we define define:

i) the *reduced E-cohomology* $\widetilde{E}^n(X) \coloneqq [\Sigma^\infty X, \Sigma^n E]$ of X and the *unreduced E-cohomology* $E^n(X) \coloneqq \widetilde{E}^n(X_+)$ of X, where X_+ denotes the space of disjoint union of X and a point.
ii) the *reduced E-homology* $\widetilde{E}_n(X) \coloneqq [\Sigma^n \mathbb{S}, \Sigma^\infty X \wedge E]$ of X and the *unreduced E-homology* $E_n(X) \coloneqq \widetilde{E}_n(X_+)$ of X.

Remark 2.12. In the above definition, we can replace the space X with a spectrum F, in which case we would obtain the definition of the E-cohomology (homology) of the spectrum F.

Remark 2.13.

i) We have $\pi_n(E) \cong [\Sigma^n \mathbb{S}, E] \cong E_n(pt) \cong E^{-n}(pt)$, and
ii) $\pi_n(F \wedge E) \cong E_n(F)$. □

We can rephrase Theorem 2.1.

Theorem 2.14 (Brown)**.** *Every generalised cohomology theory arises from a spectrum cohomology.*

Remark 2.15. Ring spectra define multiplicative cohomology theories, under Definition 2.11.

Remark 2.16. Denote by **CohThy** denotes the category of cohomology theories. Definition 2.11 gives us a functor from **HoSp** to **CohThy**. This functor is essentially surjective and full, however it is not faithful, *cf.* [Lur10, Lecture 17].

With this in mind, we will in our notation not distinguish a spectrum and the cohomology theory it represents.

Now, we establish the fact that MU together with a canonical choice of complex orientation is the universal complex oriented cohomology theory.,

Proposition 2.17. *The canonical inclusion $\Sigma^{\infty-2}\mathbb{C}\mathrm{P}^\infty = MU(1) \to MU$ determines a cohomology class in $t_{MU} \in \widetilde{MU}^2(\mathbb{C}\mathrm{P}^\infty)$. The element t_{MU} is a canonical complex orientation of MU.*

Sketch. The surjectivity is given by the inclusion $\mathbb{S} = MU(0) \to MU(1) \to MU$. □

Let (E, t) be a complex oriented cohomology theory. Given a complex vector bundle $\xi\colon X \to B$ of rank n, we can define a canonical Thom class $u_\xi \in \widetilde{E}^{2n}(\mathrm{Th}(\xi))$, *cf.* [Lur10, Lecture 5]. Thus for every $n \geq 0$, we obtain that the Thom class u_{γ_n} of the universal complex vector bundel $\gamma_n\colon EU(n) \to BU(n)$ of rank n. By definition we have

$$\begin{aligned} u_{\gamma_n} \in \widetilde{E}^{2n}(\mathrm{Th}(\gamma_n)) &= [\Sigma^\infty(\mathrm{Th}(\gamma_n)), \Sigma^{2n} E] \\ &\cong [\Sigma^{\infty-2n}\mathrm{Th}(\gamma_n), E] \\ &= [MU(n), E]. \end{aligned}$$

Thus u_{γ_n} corresponds to (the homotopy class of) a map $\phi_n\colon MU(n) \to E$, for every $n \geq 0$. Furthermore, the maps $\{\phi_n\}_{n\geq 0}$ glue nicely together: $\phi_{n-1} = \phi_n \circ \iota_n$. Therefore, we obtain a map

$$\phi\colon MU \to E.$$

Proposition 2.18. *The map ϕ is a map of ring spectra and $\phi(t_{MU}) = t$.*

Theorem 2.19. *There is a bijection*

$$\left\{\begin{matrix}\text{ring maps}\\ MU \to E\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{complex orientation}\\ \text{of the spectrum } E\end{matrix}\right\}$$
$$\phi \longmapsto \phi(t_{MU}).$$

Proof. See [Lur10, Lecture 6, Theorem 8]. □

Remark 2.20. Every complex orientation t of E determines a formal group law $f_{E,t}$. By the above theorem, in order to understand $f_{E,t}$, it is sufficient to know the corresponding ring spectrum morphism $MU \to E$ and the formal group laws $f_{MU,t_{MU}}$ that is determined by the complex orientation t_{MU}.

3. Rational version of Quillen's Theorem

The formal group law f_{MU} that is determined by t_{MU} corresponds to a map $\psi\colon L \to \pi_*MU$ (Recall that f_{MU} is a formal group law over $MU_* \cong \pi_*MU$). This map is actually an isomorphism of graded rings. The difficulty in proving this isomorphism is that homotopy groups are in general difficult to compute. Like we always do in topology, we will first compute the ordinary homology group $\mathrm{H}_*(MU;\mathbb{Z})$, *i.e.* $H\mathbb{Z}_*(MU)$, where $H\mathbb{Z}$ is the spectrum that defines the singular homology/cohomology theory.

Proposition 3.1. *Let (E,t) be a complex oriented cohomology theory. There is a canonical (with respect to choice of t) isomorphism*

$$E_*(MU) \cong \pi_*E[b_1, b_2, \dots, b_n]$$

with $\deg(b_i) = 2i$. *Especially,*

$$\mathrm{H}_*(MU;\mathbb{Z}) \cong \mathbb{Z}[b_1, b_2, \dots,].$$

Proof. See [Lur10, Lecture 7, Proposition 2]. □

The rest of the talk is to give a sketch of the following proposition.

Proposition 3.2. *The map*

$$\theta\colon L \xrightarrow{\psi} \pi_*MU \to \mathrm{H}_*(MU;\mathbb{Z})$$

is a rational isomorphism, where the second map is the Hurewicz map induced by $MU \cong MU \wedge \mathbb{S} \xrightarrow{\mathrm{id}\wedge\eta} MU \wedge H\mathbb{Z}$.

Corollary 3.3. *The map $L \to \pi_*MU$ is a rational isomorphism.*

Sketch. The Hurewicz map is always a rational isomorphism for spectra. □

The way we prove Proposition 3.2 is to study the formal group law that it classifies. To do this, let us consider the the spectrum $MU \wedge H\mathbb{Z}$. This spectrum has two complex orientations that come from the canonical orientations of MU and $H\mathbb{Z}$ respectively:

$$(*) \qquad MU(1) \xrightarrow{t_{MU}} MU \cong MU \wedge \mathbb{S} \xrightarrow{\mathrm{id}\wedge\eta} MU \wedge H\mathbb{Z}$$
$$MU(1) \xrightarrow{t_{\mathbb{Z}}} H\mathbb{Z} \cong \mathbb{S} \wedge H\mathbb{Z} \xrightarrow{\eta\wedge\mathrm{id}} MU \wedge H\mathbb{Z}.$$

This first composition maps the canonical complex orientations t_{MU} of MU to a complex orientation $t_{MU} \wedge 1$, denoted again by t_{MU}. This second composition maps the canonical complex orientations $t_{\mathbb{Z}}$ of $H\mathbb{Z}$ to a complex orientation $1 \wedge t_{\mathbb{Z}}$, denoted again by $t_{\mathbb{Z}}$.

Thus this two complex orientations gives isomorphic cohomology ring of $\mathbb{CP}^\infty$. We have

$$\mathbb{Z}[b_1, b_2, ...][[t_{\mathbb{Z}}]] \cong \pi_*(MU \wedge H\mathbb{Z})(\mathbb{CP}^\infty) \cong \mathbb{Z}[b_1, b_2, ...][[t_{MU}]],$$

because $\pi_*(MU \wedge H\mathbb{Z}) \cong \mathrm{H}_*(MU;\mathbb{Z})$.

Proposition 3.4. *Let $g(x) = x + b_1x^2 + b_2x^3 + \cdots$ be a formal power series over $\mathbb{Z}[b_1, b_2, \dots]$. We have $t_{MU} = g(t_{\mathbb{Z}})$.*

Proof. See [Lur10, Lecture 7, Claim 2]. □

Also, we have two formal group laws $f_{MU\wedge H\mathbb{Z},t_{MU}}$ and $f_{MU\wedge H\mathbb{Z},t_{\mathbb{Z}}}$ over $\mathbb{Z}[b_1, b_2, \dots]$ determined by the complex orientations. More concretely,

$$\mu^*(t_{\mathbb{Z}}) = f_{MU\wedge H\mathbb{Z},t_{\mathbb{Z}}}(\mathrm{pr}_1^*\, t_{\mathbb{Z}}, \mathrm{pr}_2^*\, t_{\mathbb{Z}}) = \mathrm{pr}_1^*\, t_{\mathbb{Z}} + \mathrm{pr}_2^*\, t_{\mathbb{Z}}$$
$$\mu^*(t_{MU}) = f_{MU\wedge H\mathbb{Z},t_{MU}}(\mathrm{pr}_1^*\, t_{MU}, \mathrm{pr}_2^*\, t_{MU}),$$

where μ is the multiplication of $\mathbb{CP}^\infty$.

Comparing the Hurewicz homomorphism with the composition $(*)$, we see that the map θ classifies the formal group law $f_{MU\wedge H\mathbb{Z},t_{MU}}$.

Corollary 3.5. *We have* $f_{MU\wedge H\mathbb{Z},t_{MU}}(x,y) = g(g^{-1}(x)+g^{-1}(y))$*.*

Proof. We have

$$\begin{aligned} f_{MU\wedge H\mathbb{Z},t_{MU}}(\mathrm{pr}_1^* t_{MU}, \mathrm{pr}_2^* t_{MU}) &= \mu^*(t_{MU}) \\ &= \mu^*(g(t_{\mathbb{Z}})) \\ &= g(\mu^*(t_{\mathbb{Z}})) \\ &= g(f_{MU\wedge H\mathbb{Z},t_{\mathbb{Z}}}(\mathrm{pr}_1^* t_{\mathbb{Z}}, \mathrm{pr}_2^* t_{\mathbb{Z}})) \\ &= g(\mathrm{pr}_1^* t_{\mathbb{Z}} + \mathrm{pr}_2^* t_{\mathbb{Z}}) \\ &= g\left(\mathrm{pr}_1^*\left(g^{-1}(t_{MU})\right) + \mathrm{pr}_2^*\left(g^{-1}(t_{MU})\right)\right) \\ &= g\left(g^{-1}\left(\mathrm{pr}_1^*\left(t_{MU}\right)\right) + g^{-1}\left(\mathrm{pr}_2^*\left(t_{MU}\right)\right)\right) \end{aligned}$$

□

Thus θ classifies the formal group law $g\left(g^{-1}(x)+g^{-1}(y)\right)$. By [Lur10, Lecture 2, Lemma 10], we know that θ is an rational isomorphism.

Remark 3.6. We can generalise the above argument/construction. In other words, we can replace $H\mathbb{Z}$ by any other complex oriented cohomology theory (E, t_E), then we would obtain the isomorphism

$$f_{MU\wedge E,t_{MU}}(x,y) = g \circ f_{MU\wedge E,t_E}(g^{-1}(x)+g^{-1}(y)).$$

References

[Ada74] J. F. Adams. *Stable homotopy and generalised homology.* 1974.

[Lur10] J. Lurie. "Chromatic homotopy theory" (2010). URL: http://www.math.harvard.edu/~lurie/252x.html.

[Mal14] C. Malkiewich. "The stable homotopy category" (2014). URL: http://people.math.binghamton.edu/malkiewich//stable.pdf.