Galois Symmetries of Knot Spaces

following the newest preprint by Boavida de Brito–Horel

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Goal

Present the following idea of the paper [BH]:

Use the Grothendieck–Teichmüller group action on the p-completed E_n-operads, to show that a lot of differentials vanishes, in the p-local Bousfield–Kan spectral sequence associated to the manifold calculus tower of the space of knots.

The authors use these computations to deduce that some classical knot invariants produced from the manifold calculus tower are universal Vassiliev invariants, over $\mathbb{Z}_{(p)}$.

Outline

Long knots

Definition

Fix a smooth embedding $c\colon \mathbb{R}^1 \to \mathbb{R}^d$, $t \mapsto (t, 0, \ldots, 0)$. The *space* $\mathcal{K}^d$ *long knots* is the space of smooth embeddings $\mathbb{R}^1 \to \mathbb{R}^d$ that coincides with c in the complement of $(0, 1)$, equipped with Whitney topology. An element $K \in \mathcal{K}$ is called a *long knot*.

Figure: An example of a long knot in $\mathbb{R}^3$

Manifold calculus tower of $\mathcal{K}^d$

Apply the theory of manifold calculus to the functor

$$\mathsf{Emb}_c(-,\mathbb{R}^d)\colon \mathbf{Open}(\mathbb{R})^{\mathrm{op}} \to \mathbf{Top}$$

and evaluate it at $\mathbb{R}$ gives us

$$\begin{array}{ccccccccc} & & \mathcal{K} & & & & & & \\ & \swarrow & \downarrow{\scriptstyle \eta_n} & \searrow^{\eta_{n-1}} & & & \searrow^{\eta_0} & & \\ \cdots & \xrightarrow[r_{n+1}]{} & \mathsf{T}_n(\mathbb{R}^1,d) & \xrightarrow[r_n]{} & \mathsf{T}_{n-1}(\mathbb{R}^1,d) & \xrightarrow[r_{n-1}]{} & \cdots & \xrightarrow[r_1]{} & \mathsf{T}_0(\mathbb{R}^1,d). \end{array}$$

where $\mathsf{T}_n(-,d)$ is the best approximation of $\mathsf{Emb}_c(-,\mathbb{R}^d)$ by n-excisive functors.

A result from [BH]

1. Drop the dimension d from notation.
2. Recall the tower T of fibrations

$$\cdots \to \mathrm{T}_n(\mathbb{R}^1) \to \mathrm{T}_{n-1}(\mathbb{R}^1) \to \cdots \to \mathrm{T}_0(\mathbb{R}^1).$$

associated to $\mathcal{K}^d$. Denote by $\{E^r_{-s,t}(T)\}_{s\leq 0, t\geq 0}$ the associated homotopy Bousfield Kan spectral sequence.

Theorem ([BH, Theorem B])

Let p be a prime. The differential $E^r_{-s,t}(T) \to E^r_{-s-r,t+r-1}(T)$ vanishes p-locally if $r-1$ is not a multiple of $(p-1)(d-2)$ and if $t < 2p-2+(s-1)(d-2)$.

Two consequences

Theorem ([BH, Theorem A])

Let p be a prime, $d = 3$ and $n \leq p + 1$. Then the canonical map $\pi_0(\mathcal{K}^3) \to$ is a universal Vassiliev invariant of degree n over $\mathbb{Z}_{(p)}$

[BH, Theorem C]

One can also calculate the p-local homotopy group of $\mathrm{T}_n(\mathbb{R}^1, d)$ in certain range.

Grothendieck–Teichmüller group action on $\mathbb{Z}_p$-modules

1. Let p be a prime, and consider the Grothendieck–Teichmüller group GT_p. This is a pro p-complete group.
2. There is a surjective group homomorphism $\chi\colon \mathrm{GT}_p \to \mathbb{Z}_p^\times$, called the *cyclotomic character*.
3. For every $\mathbb{Z}_p$-module M and for $n \geq 0$, we can equip M with a *cyclotomic* GT_p*-action of weight* n by setting $\gamma.m := \chi(\gamma)^n m$, for $\gamma \in \mathrm{GT}_p$ and $m \in M$.
4. Let $0 \to M' \to M \to M'' \to 0$ be a s.e.s of $\mathbb{Z}_p$-modules that is GT_p-equivariant. If the GT_p-action on M is of weight n, then so is the case of the GT_p action on M' and M.
5. Let $f\colon M \to N$ be a GT_p-equivariant $\mathbb{Z}_p$-module homomorphism. The action of GT_p on M and N are m and n respectively. Then f is null if $m - n$ is not a multiple of $p - 1$.

Key ingredient for the proof of [BH, Theorem B]

Theorem ([BH, Theorem 2.3])

Let $d \geq 3$. There is a GT_p-action on the p-completion of little d-disk operad $\mathsf{L}_p \, \mathrm{E}_d$. The induced action on $\pi_{d-1}(\mathsf{L}_p \, \mathrm{E}_d(2))$ is a cyclotomic action of weight 1.

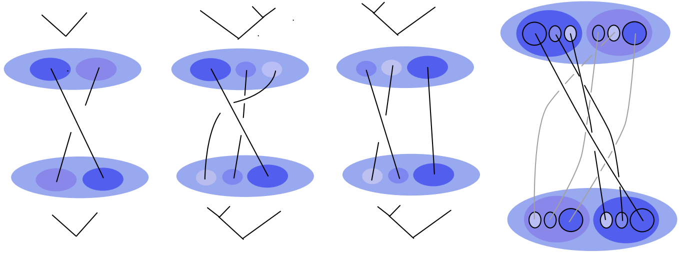
Theorem ([BH, Theorem B])

Let p be a prime. The differential $E^r_{s,t}(T) \to E^r_{s-r,t+r-1}(T)$ vanishes p-locally if $r-1$ is not a multiple of $(p-1)(d-2)$ and if $t < 2p-2+(s-1)(d-2)$.

Parenthesised braid operad

Definition

Let $n \in \mathbb{N}$. Define the groupoid $\mathrm{PaB}(n)$ whose objects are binary trees with n leaves labeled from 1 to n. A morphism of two tress $T_1 \to T_2$ is a braid with n-strands connecting the i-th leaf in T_1 to the i-th leaf in T_2.
The operad PaB of *parenthesised braid* is an operad in the **Grpd** with $\mathrm{PaB}(n)$ the n-th groupoid.



One definition of GT_p

Definition

The Grothendieck–Teichmüller group GT_p is the group of automorphisms of $\mathrm{PaB}_p^\wedge$ that fix the objects, where $(-)_p^\wedge$ denotes the pro p-completion.

Theorem

There is an induced action

$$\mathrm{GT}_p \to \mathsf{Aut}^h\left(\mathrm{B}\left(\mathrm{PaB}_p^\wedge\right)\right) \cong \mathsf{Aut}^h\left((\mathrm{BPaB})_p^\wedge\right) \cong \mathsf{Aut}^{\mathsf{h}}((\mathrm{E}_2)_p^\wedge),$$

where the isomorphisms are induced by equivalence of ∞-operads $\mathrm{B}\left(\mathrm{PaB}_p^\wedge\right) \simeq (\mathrm{BPaB})_p^\wedge \simeq (\mathrm{E}_2)_p^\wedge$ *in pro p-completed spaces.*

Sketch of the proof of [BH, Theorem 2.3]

Theorem ([BH, Theorem 2.3])

Let $d \geq 3$. There is a GT_p-action on the p-completion of little d-disk operad $\mathsf{L}_p \mathrm{E}_d$. The induced action on $\pi_{d-1}(\mathsf{L}_p \mathrm{E}_d(2))$ is a cyclotomic action of weight 1.

1. Using Dunn additivity, GT_p acts on $(\mathrm{E}_d)_p^\wedge$ via acting on the $(\mathrm{E}_2)_p^\wedge$ component via the previous theorem and acting on the rest $(\mathrm{E}_{d-2})_p^\wedge$ component trivially.
2. For $d \geq 3$ and $k \geq 0$, the space $\mathrm{E}_d(k)$ is nilpotent and of finite $\mathbb{F}_p$-type. Thus we have $\mathsf{L}_p(E_d) \simeq \mathrm{holim}(\mathrm{E}_d)_p^\wedge$. Therefore, GT_p also acts on $\mathsf{L}_p(\mathrm{E}_d)$.
3. The compositions

$$\mathrm{GT}_p \to \mathrm{Aut}^h(\mathsf{L}_p \mathrm{E}_d) \to \mathrm{Aut}^h(\mathsf{L}_p \mathrm{E}_d(2)) \cong \mathbb{Z}_p^\times$$

coincides with the cyclotomic character.

Recall: A result from [BH]

- Recall the manifold calculus tower T of fibrations

$$\cdots \to \mathsf{T}_n(\mathbb{R}^1) \to \mathsf{T}_{n-1}(\mathbb{R}^1) \to \cdots \to \mathsf{T}_0(\mathbb{R}^1).$$

associated to the space $\mathcal{K}^d$ of smooth embeddings $\mathbb{R}^1 \to \mathbb{R}^d$. Denote by $\{E^r_{-s,t}(T)\}_{s\leq 0, t\geq 0}$ the associated homotopy Bousfield Kan spectral sequence.

Theorem ([BH, Theorem B])

Let p be a prime. The differential $E^r_{-s,t}(T) \to E^r_{s-r,t+r-1}(T)$ vanishes p-locally if $r-1$ is not a multiple of $(p-1)(d-2)$ and if $t < 2p-2+(s-1)(d-2)$.

- Introduce how the proof of this theorem goes by using the GT_p-action on $\mathsf{L}_p\,\mathrm{E}_d$.

How does T relates to mapping space of operads?

- Recall $T\colon \cdots \to \mathsf{T}_n(\mathbb{R}^1) \to \mathsf{T}_{n-1}(\mathbb{R}^1) \to \cdots \to \mathsf{T}_0(\mathbb{R}^1)$.
- (Boavida de Brito–Weiss) $\mathsf{T}_n(\mathbb{R}^1)$ fits into a homotopy fibre sequence

$$\mathsf{T}_n(\mathbb{R}^1) \longrightarrow \Omega\,\mathsf{Map}^{\mathsf{h}}_{\leq 2}(\mathrm{E}_1, \mathrm{E}_d) \overset{\Omega\, g_n}{\longrightarrow} \Omega\,\mathsf{Map}^{\mathsf{h}}_{\leq n}(\mathrm{E}_1, \mathrm{E}_d)\ .$$

  Thus $\mathsf{T}_n(\mathbb{R}^1) \simeq \Omega^2 \,\mathrm{hofib}(g_n)$ with g_n the map induced by truncation.
- Rewrite T as the tower

$$\cdots \to \Omega^2 \,\mathrm{hofib}(g_n) \to \Omega^2 \,\mathrm{hofib}(g_{n-1}) \to \cdots$$

p-completion of T

- Recall T is $\cdots \to \Omega^2 \operatorname{hofib}(g_n) \to \Omega^2 \operatorname{hofib}(g_{n-1}) \to \cdots$ with $g_n \colon \mathsf{Map}^{\mathsf{h}}_{\leq k}(\mathrm{E}_1, \mathrm{E}_d) \xrightarrow{\mathrm{trun}} \mathsf{Map}^{\mathsf{h}}_{\leq 2}(\mathrm{E}_1, \mathrm{E}_d)$
- Apply p-completion L_p, we obtain a tower $T(\mathbb{Z}_p)$:

$$\cdots \to \Omega^2 \mathsf{L}_p \operatorname{hofib}(g_n) \to \Omega^2 \mathsf{L}_p \operatorname{hofib}(g_{n-1}) \to \cdots$$

- Apply L_p inside the mapping space, we obtain a tower $T'(\mathbb{Z}_p)$

$$\cdots \to \Omega^2 \operatorname{hofib}(g'_n) \to \Omega^2 \operatorname{hofib}(g'_{n-1}) \to \cdots ,$$

  with $g'_n \colon \mathsf{Map}^{\mathsf{h}}_{\leq k}(\mathrm{E}_1, \mathsf{L}_p \mathrm{E}_d) \xrightarrow{\mathrm{trun} \circ \mathsf{L}_p} \mathsf{Map}^{\mathsf{h}}_{\leq 2}(\mathrm{E}_1, \mathsf{L}_p \mathrm{E}_d)$
- The towers $T(\mathbb{Z}_p)$ and $T'(\mathbb{Z}_p)$ are weak homotopy equivalent, by analysing the layers.

The associated Bousfield–Kan spectral sequences

- Three towers T, $T(\mathbb{Z}_p)$ and $T'(\mathbb{Z}_p)$ where $T(\mathbb{Z}_p) \simeq T'(\mathbb{Z}_p)$.
- For $r \geq 0$, the map $E^r_{-s,t}(T) \to E^r_{-s,t}(T(\mathbb{Z}_p))$ is given by tensoring with $\mathbb{Z}_p$.
- The Grothendieck–Teichmüller group GT_p acts on $T'(\mathbb{Z}_p)$.
- The associated spectral sequence $\{E^r_{s,t}(T(\mathbb{Z}_p))\}_{r\geq 0}$ is also equipped an action of GT_p, induced from the action on $\mathsf{L}_p\,\mathrm{E}_d$.

$$T\colon \cdots \to \Omega^2 \operatorname{hofib}(g_n) \to \Omega^2 \operatorname{hofib}(g_{n-1}) \to \cdots$$

$$T(\mathbb{Z}_p)\colon \cdots \to \Omega^2 \mathsf{L}_p \operatorname{hofib}(g_n) \to \Omega^2 \mathsf{L}_p \operatorname{hofib}(g_{n-1}) \to \cdots$$

$$T'(\mathbb{Z}_p)\colon \cdots \to \Omega^2 \operatorname{hofib}(g'_n) \to \Omega^2 \operatorname{hofib}(g'_{n-1}) \to \cdots ,$$

with $g_n\colon \mathsf{Map}^{\mathsf{h}}_{\leq k}(\mathrm{E}_1, \mathrm{E}_d) \to \mathsf{Map}^{\mathsf{h}}_{\leq 2}(\mathrm{E}_1, \mathrm{E}_d)$ and $g'_n = \mathsf{L}_p(g_n)$.

GT_p action on the spectral sequence

- Recall that for every $\mathbb{Z}_p$-module M and for $n \geq 0$, we can equip M with a cyclotomic GT_p-action of weight n by setting $\gamma.m := \chi(\gamma)^n m$, for $\gamma \in \mathrm{GT}_p$ and $m \in M$.

The following is a crucial step in the proof of [BH, Theorem B].

Theorem ([BH, Theorem 4.3 over $\mathbb{Z}_p$])

Let $d \geq 3$ and $t < 2p - 2 + (s-1)(d-2)$.

1. *For $t = n(d-2) + 1$ with $n \geq (s-1)$, the GT_p-action on $E^1_{-s,t}$ is of weight n.*
2. *Otherwise, $E^1_{-s,t}(T'(\mathbb{Z}_p))$ is 0.*

Theorem B follows

Corollary ([BH, Theorem 4.1])

Let p be a prime. The differential $d^r : E^r_{-s,t}(T) \to E^r_{s-r,t+r-1}(T)$ vanishes over $\mathbb{Z}_p$ if $r-1$ is not a multiple of $(p-1)(d-2)$ and if $t < 2p-2+(s-1)(d-2)$.

- Recall that the weight of a GT_p action on $\mathbb{Z}_p$-modules are preserved under taking submodules and quotients. Thus for $t = n(d-2)+1$, the GT_p-action on $E^r_{-s,t}(T(\mathbb{Z}_p))$ is of weight n.
- The map d^r is GT_p-equivariant. Thus $d^r = 0$ if the difference of the weights on the source and target is not a multiple of $p-1$.

Therefore, [BH, Theorem B], replacing $\mathbb{Z}_p$ by $\mathbb{Z}_{(p)}$ in the above corollary, follows. Since

$$E^r_{-s,t}(T(\mathbb{Z}_p)) \cong E^r_{-s,t}(T) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \left(E^r_{-s,t}(T) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}\right) \otimes_{\mathbb{Z}(p)} \mathbb{Z}_p,$$

and a morphism f of $\mathbb{Z}_{(p)}$-modules is 0 iff $f \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p = 0$

Proof of the crucial step

Theorem ([BH, Theorem 4.3 over $\mathbb{Z}_p$])

Let $d \geq 3$ and $t < 2p - 2 + (s-1)(d-2)$.

1. *For $t = n(d-2) + 1$ with $n \geq (s-1)$, the GT_p-action on $E^1_{-s,t}$ is of weight n.*
2. *Otherwise, $E^1_{-s,t}(T'(\mathbb{Z}_p))$ is 0.*

- First of all, we need to understand the group $E^1_{-s,t}(T'(\mathbb{Z}_p))$ more concretely.
- (Bousfield–Kan),

$$E^1_{-s,t}(T'(\mathbb{Z}_p)) \cong \pi_{t-s}(F^s) \otimes \mathbb{Z}_p,$$

  where F^s is the s-th layer in the tower $T'(\mathbb{Z}_p)$.
- So we need to understand what is F^s.

Analyse $E^1_{-s,t}(T'(\mathbb{Z}_p))$

Recall that $T'(\mathbb{Z}_p)\colon \cdots \to \Omega^2 \operatorname{hofib}(g'_n) \to \Omega^2 \operatorname{hofib}(g'_{n-1}) \to \cdots$
with $g_n\colon \mathsf{Map}^{\mathsf{h}}_{\leq k}(\mathrm{E}_1, \mathsf{L}_p\, \mathrm{E}_d) \to \mathsf{Map}^{\mathsf{h}}_{\leq 2}(\mathrm{E}_1, \mathsf{L}_p\, \mathrm{E}_d)$

We have

$$E^1_{-s,t}(T'(\mathbb{Z}_p)) \cong \pi_t\, \mathsf{thofib}\left(S \subseteq \underline{s} \mapsto \mathsf{L}_p\, \mathsf{Emb}(S, \mathbb{R}^d)\right) \otimes \mathbb{Z}_p$$

for $s > 2$ and 0 otherwise. Here, $\underline{s} := \{1, 2, \cdots, s\}$.

This is because

Theorem (Göppl)

Let P ba an operad with weakly contractible $P(0)$ and $P(1)$. For $n \geq 2$, there is a homotopy fibre sequence

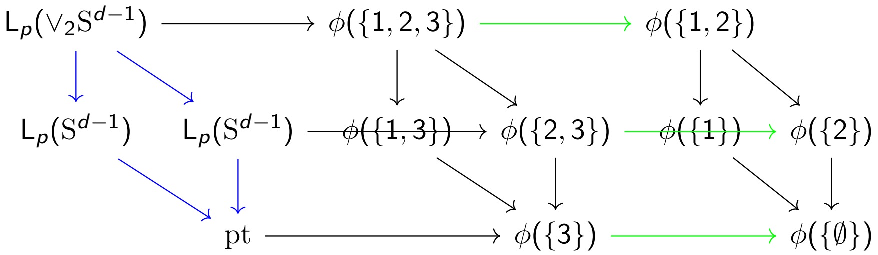
$$\Omega^{n-2}\, \mathsf{thofib}\,(S \subseteq \underline{n} \mapsto P(S)) \to \mathsf{Map}^{\mathsf{h}}(\mathrm{E}_1, P)_{\leq n} \to \mathsf{Map}^{\mathsf{h}}(\mathrm{E}_1, P)_{\leq n-1}$$

thofib $\left(S \subseteq \underline{s} \mapsto \mathrm{L}_p \, \mathrm{Emb}(S, \mathbb{R}^d)\right)$

Denote the cube $S \subseteq \underline{s} \mapsto \mathrm{L}_p \, \mathrm{Emb}(S, \mathbb{R}^d)$ by ϕ. Note that $\phi(S)$ is the p-completion of configuration space of $\sharp S$ points in $\mathbb{R}^d$.

- Then $\mathrm{thofib}(\phi) = \mathrm{thofib}(R \subseteq \underline{s-1} \mapsto \mathrm{L}_p \vee_R \mathrm{S}^{d-1})$.
- By calculation, the canonical map $\mathrm{thofib}(\phi) \to \mathrm{L}_p(\vee_{s-1}\mathrm{S}^{d-1})$ induces injections on homotopy groups.

An illustration for $s = 3$:



Recall: the crucial step

Theorem ([BH, Theorem 4.3 over $\mathbb{Z}_p$])

Let $d \geq 3$ and $t < 2p - 2 + (s-1)(d-2)$.

1. *For $t = n(d-2)+1$ with $n \geq (s-1)$, the GT_p-action on $E^1_{-s,t}$ is of weight n.*
2. *Otherwise, $E^1_{-s,t}(T'(\mathbb{Z}_p))$ is 0.*

- We have $E^1_{-s,t} \cong \pi_t\left(\mathrm{thofib}(R \subseteq \underline{s-1} \mapsto \mathsf{L}_p \vee_R \mathrm{S}^{d-1})\right) \otimes \mathbb{Z}_p$.
- $\pi_t\left(\mathrm{thofib}(R \subseteq \underline{s-1} \mapsto \mathsf{L}_p \vee_R \mathrm{S}^{d-1})\right)$ maps injectively to $\pi_t(\mathsf{L}_p \vee_{s-1} \mathrm{S}^{d-1}))$

Proof

- We have

$$E^1_{-s,t}(T'(\mathbb{Z}_p)) \cong \pi_t(\mathrm{thofib}(\phi)) \otimes \mathbb{Z}_p \subseteq \pi_t(\vee_{s-1}\mathrm{S}^{d-1}) \otimes \mathbb{Z}_p$$

- The rational homotopy group of $\vee_{s-1}\mathrm{S}^d$ concentrated in degree $n(d-2)+1$ for $n \geq s-1$, by Hilton–Milnor theorem.
- The homotopy group $\pi_t(\vee_{s-1}\mathrm{S}^d)$ is p-torsion free for $n \leq (s-1)(d-2)+2p-2$, again by Hilton–Milnor and Serre. Therefore, for t in this range, $E^1_{-s,t}(T'(\mathbb{Z}_p))$ coincide with $\pi_t(\vee_{s-1}\mathrm{S}^{d-1}) \otimes \mathbb{Q}_p$, and thus 0 for $t \neq n(d-2)+1$ with $n \geq (s-1)$.
- The GT_p-action on $\pi_{d-1}(\mathsf{L}_p(\vee_k\mathrm{S}^{d-1}))$ is cyclotomic of weight 1, for $k \geq 1$, *cf.* [BH, Proposition 4.4].
- Thus, the GT_p-action on $\pi_{n(d-2)+1}(\vee_k\mathrm{S}^{d-1})$ is of weight n, since an element in $\pi_{n(d-2)+1}$ is give by sums of Whitehead products of of length n.

Summary

- We introduced the Grothendieck–Teichüller group, via the parenthesised braid operads, and see that GT_p acts on the p-completed E_d operads, for $d \geq 2$.
- We see the relation between the tower T of fibrations produced by manifold calculus of the space of long knots in $\mathbb{R}^d$, and a tower of fibrations of derived mapping space of E_n-operads.
- We use this relation to give a GT_p-action on the homotopy Bousfield–Kan spectral sequence associated to the p-complete tower $T(\mathbb{Z}_p)$.
- By considering the differences of the weight of the GT_p-action on the target and the source of a differential, we showed that the differential vanishes p-locally in certain given range.
- There are more applications and consequences in [BH].

References

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