The Frobenius and the Tate diagonal

eCHT Reading Seminar on THH

Yuqing Shi

University of Utrecht

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Outline

Motivation from Algebra

Let A be an abelian group. The diagonal map

$$\Delta_{\mathrm{C}_p}\colon A \to \left(A^{\otimes p}\right)^{\mathrm{C}_p},\ a \mapsto a \otimes a \otimes \cdots \otimes a$$

is not a group homomorphism. One way to fix this:

Definition

Let M be an abelian group with finite group G-action. The *norm map* is

$$\mathrm{Nm}_G(M)\colon M_G \to M^G,\ m \mapsto \sum_{g \in G} g.m$$

Motivation from Algebra

Proposition

The composition

$$\Delta_p \colon A \xrightarrow{\Delta_{\mathrm{C}_p}} \left(A^{\otimes p}\right)^{\mathrm{C}_p} \twoheadrightarrow \left(A^{\otimes p}\right)^{\mathrm{C}_p} \Big/ \mathrm{Nm}_{\mathrm{C}_p}\left(A^{\otimes p}\right)$$

is a homomorphism of abelian groups.
Moreover, the target of this map is p-torsion, and the induced map

$$A/p \to \left(A^{\otimes p}\right)^{\mathrm{C}_p} \Big/ \mathrm{Nm}_{\mathrm{C}_p}\left(A^{\otimes p}\right)$$

is an isomorphism.

Tate diagonal for spectra

Recall that for a spectrum $Y \in \mathbf{Sp}^G$ with G a finite group. We have a cofiber sequence

$$Y_{\mathrm{h}G} \xrightarrow{\mathsf{Nm}_G(Y)} Y^{\mathrm{h}G} \to Y^{\mathrm{t}G}.$$

The analogues statements in $\mathbf{Sp}$:

Theorem ([NS18, Theorem III.1.7, Proposition III.3.1])

For every $X \in \mathbf{Sp}$, there is a unique map

$$\Delta_p(X)\colon X \to \left(X^{\otimes_{\mathbb{S}} p}\right)^{\mathrm{tC}_p} \in \mathbf{Sp}$$

such that it is natural in X and is symmetric monoidal. If X is bounded below, then $(X^{\otimes p})^{\mathrm{tC}_p}$ is weak equivalent to the p-completion of X.

Tate diagonal for spectra

Definition

For every $X \in \mathbf{Sp}$, we call the map $\Delta_p(X)$ the *Tate diagonal*.

Remark

Let R be an E_n-ring spectra, with $0 \leq n \leq \infty$. The map $\Delta_p(R)$ is a map of E_n-ring spectra, because $\Delta_p(-)$ is symmetric monoidal.

Example

For the sphere spectrum $\mathbb{S}$, the statement that the map $\Delta_p(\mathbb{S})\colon \mathbb{S} \to \mathbb{S}^{t\mathrm{C}_p}$ is a p-completion is equivalent to the Segal conjecture.

The Frobenius

- Use the Tate diagonals to construct extra structures on the spectra $\mathsf{THH}(R)$, for R an E_n-ring spectrum with $1 \leq n \leq \infty$.
- For simplicity, we do this in detail for E_∞-ring spectra and sketch the case for E_1-ring spectra.

A universal property of $\mathsf{THH}(R)$ for R an E_∞-ring spectra

Recall fron last talk that there is a natural $\mathbb{T}$-action on $\mathsf{THH}(R)$, which is compatible with the lax symmetric monoidal structure on the functor $\mathsf{THH}(-)$.

Proposition (McClure–Schwänzl–Vogt)

Let R be an E_∞-ring spectrum. The canonical map $R \xrightarrow{\iota} \mathsf{THH}(R)$ induces an equivalence

$$\mathsf{Map}^{\mathbb{T}}_{\mathrm{E}_\infty}(\mathsf{THH}(R), Z) \xrightarrow{\iota^*} \mathsf{Map}_{\mathrm{E}_\infty}(R, Z)$$

for every $Z \in \mathbf{Alg}^{\mathbb{T}}_{\mathrm{E}_\infty}(\mathbf{Sp})$.

Sketch of the proof of MSV

It suffices to see a natural isomorphism

$$\mathsf{THH}(R) \simeq R \otimes \mathbb{T} \in \mathbf{Sp}_{\mathrm{E}_\infty}.$$

Indeed, the adjunction

$$Fr : \mathbf{Alg}_{\mathrm{E}_\infty}(\mathbf{Sp}) \underset{\text{forget}}{\overset{-\otimes\mathbb{T}}{\rightleftarrows}} \mathbf{Alg}^{\mathbb{T}}_{\mathrm{E}_\infty}(\mathbf{Sp}) : U$$

implies that the canonical map $R \to R \otimes \mathbb{T}$ is initial among all the E_∞-ring maps from R to $Z \in \mathbf{Alg}^{\mathbb{T}}_{\mathrm{E}_\infty}(\mathbf{Sp})$.

Sketch the equivalence $\mathrm{THH}(R) \simeq R \otimes \mathrm{S}^1$

- $R \otimes \mathbb{T} := \operatorname{colim}_{\mathbb{T}} R$, where the colimit is taken in the ∞-category $\mathbf{Alg}_{\mathrm{E}_\infty}(\mathbf{Sp})$.
- The standard simplicial model $(\Delta^1/\partial\Delta^1)_\bullet$ for the circle has $n+1$ simplices in dimension n.
- $R \otimes \mathbb{T} \simeq |R \otimes (\Delta^1/\partial\Delta^1)_\bullet| \simeq \mathrm{THH}(R)$. The first weak equivalence is [MSV97, Proposition 4.3]. For the second equivalence, we use the fact that the coproduct in $\mathbf{Alg}_{\mathrm{E}_\infty}(\mathbf{Sp})$ is the tensor product.

The Frobenius for E_∞-ring spectra

- Analogously, the canonical map $R \to \mathsf{THH}(R)$ induces an E_∞-ring map
$$R \otimes \mathrm{C}_p \to \mathsf{THH}(R)$$
which is C_p-equivariant.Note that $R \otimes \mathrm{C}_p \simeq R^{\otimes_{\mathbb{S}} p}$.
- Applying the Tate construction and precompose with the Tate diagonal map, we obtain a morphism
$$R \xrightarrow{\Delta_p} (R \otimes \mathrm{C}_p)^{\mathrm{tC}_p} \to \mathsf{THH}(R)^{\mathrm{tC}_p}$$
of E_∞-ring spectra.
- Note that $\mathsf{THH}(R)^{\mathrm{tC}_p}$ has a residue $\mathbb{T}/\mathrm{C}_p \cong \mathbb{T}$ action.

The Frobenius for E_∞-ring spectra

Definition

Let R be an E_∞-ring spectrum. The *Frobenius* of $\mathrm{THH}(R)$ is the unique $\mathbb{T}$-equivariant, E_∞-ring spectra morphism φ_p, such that the diagram

$$\begin{tikzcd} R \arrow[r, "\iota"] \arrow[d, "\Delta_p"] & \mathrm{THH}(R) \arrow[d, dashed, "\varphi_p"] \\ (R \otimes \mathrm{C}_p)^{\mathrm{tC}_p} \arrow[r] & \mathrm{THH}(R)^{\mathrm{tC}_p} \end{tikzcd}$$

Sketch: The Frobenius for E_1-ring spectra

Let R be a E_1-ring spectrum.

$$
\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & R^{\otimes_{\mathbb{S}} 3} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, dashed, "\Delta_p"] & R^{\otimes_{\mathbb{S}} 2} \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d, dashed, "\Delta_p"] & R \arrow[d, "\Delta_p(R)"] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \left(R^{\otimes_{\mathbb{S}} 3p}\right)^{\mathrm{tC}_p} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \left(R^{\otimes_{\mathbb{S}} 2p}\right)^{\mathrm{tC}_p} \arrow[r, shift left=1] \arrow[r, shift right=1] & \left(R^{\otimes_{\mathbb{S}} p}\right)^{\mathrm{tC}_p}
\end{tikzcd}
$$

- The colimit of the upper simplicial ring spectrum is $\mathrm{THH}(R)$.
- The colimit of the lower one is $\mathrm{THH}(R)^{\mathrm{tC}_p}$.
- We can extend the Tate diagonal $\Delta_p(R)$ for R to a map from the upper to the lower simplicial ring spectra.
- See [NS18, Section III.2] for details.

The Tate construction and the p-completion

In this section we want to prove the following

Lemma ([NS18, Lemma II.4.2])

Let X be a bounded below spectrum with $\mathbb{T}$-action. The canonical map

$$X^{\mathrm{t}\mathbb{T}} \to \left(X^{\mathrm{t}C_p}\right)^{\mathrm{h}(\mathbb{T}/C_p)}$$

is a p-completion.

$$\begin{tikzcd}
\Sigma X_{\mathrm{h}\mathbb{T}} \simeq \left(X_{\mathrm{h}C_p}\right)_{\mathrm{h}\mathbb{T}/C_p} \arrow[r, "\mathsf{Nm}"] \arrow[d, "\mathsf{Nm}"] & X^{\mathrm{h}\mathbb{T}} \arrow[r] \arrow[d] & X^{\mathrm{t}\mathbb{T}} \arrow[d, red] \\
\Sigma \left(X_{\mathrm{h}C_p}\right)^{\mathrm{h}\mathbb{T}/C_p} \arrow[r, "\mathsf{Nm}"] & \left(X^{\mathrm{h}C_p}\right)^{\mathrm{h}\mathbb{T}/C_p} \arrow[r] & \left(X^{\mathrm{t}C_p}\right)^{\mathrm{h}\mathbb{T}/C_p}
\end{tikzcd}$$

Direct consequences

Let $R \in \mathbf{Alg}_{\mathrm{E}_1}^{\geq 0}(\mathbf{Sp})$. Then $\mathrm{THH}(R)$ satisfies the hypothesis of the above lemma. The maps

$$\mathrm{THH}(R)^{\mathrm{h}\mathbb{T}} \xrightarrow{\varphi_p^{\mathrm{h}\mathbb{T}}} \left(\mathrm{THH}(R)^{\mathrm{tC}_p}\right)^{\mathrm{h}\mathbb{T}} \overset{\text{Lemma}}{\simeq} \left(\mathrm{THH}(R)^{\mathrm{t}\mathbb{T}}\right)_p^{\wedge},$$

for all primes p, induce a map

$$\varphi\colon \mathrm{THH}(R)^{\mathrm{h}\mathbb{T}} \to \left(\mathrm{THH}(R)^{\mathrm{t}\mathbb{T}}\right)^{\wedge} := \prod_p \left(\mathrm{THH}(R)^{\mathrm{t}\mathbb{T}}\right)_p^{\wedge},$$

where $(-)^{\wedge}$ denotes the profinite completion.
Furthermore, the map φ fits into a fibre sequence

$$\mathrm{TC}(R) \to \mathrm{TC}(R)^{-} \xrightarrow{\varphi} \mathrm{TP}(R)^{\wedge},$$

see [NS18, Remark II.4.3].

Proof of the Lemma

Lemma

Let X be a bounded below spectrum with $\mathbb{T}$-action. The canonical map

$$X^{\mathrm{t}\mathbb{T}} \to \left(X^{\mathrm{tC}_p}\right)^{\mathrm{h}(\mathbb{T}/\mathrm{C}_p)}$$

is a p-completion.

The proof has two main steps:

1. Reduce to the case where $X = \mathrm{HM}$ with trivial $\mathbb{T}$-action, where M is a torsion free abelian group.
2. Compare the HFPSS's of $\left(\mathrm{HM}^{\mathrm{hC}_p}\right)^{\mathrm{h}\,\mathbb{T}/\mathrm{C}_p}$ and $\left(\mathrm{HM}^{\mathrm{tC}_p}\right)^{\mathrm{h}\,\mathbb{T}/\mathrm{C}_p}$

Reduction to HM with M torsion free abelian

Reduce to the case of Eilenberg–MacLane spectra:

- Let $Y \in \mathbf{Sp}^G$. We have commutative diagram:

$$\begin{tikzcd}
Y_{\mathrm{h}G} \arrow[r] \arrow[d, "\simeq"] & Y^{\mathrm{h}G} \arrow[r] \arrow[d, "\simeq"] & Y^{\mathrm{t}G} \arrow[d, "\simeq"] \\
\varprojlim (\tau_{\leq n} Y)_{\mathrm{h}G} \arrow[r] & \varprojlim (\tau_{\leq n} Y)^{\mathrm{h}G} \arrow[r] & \varprojlim (\tau_{\leq n} Y)^{\mathrm{t}G}
\end{tikzcd}$$

- Limits of p-complete objects are p-complete.

Reduction to the case HM with M a torsion free abelian group

- X^{tC_p} is p-torsion. In particular, $\left(X^{\mathrm{tC}_p}\right)^{\mathrm{h}\,\mathbb{T}/\mathrm{C}_p}$ is p-complete
- Use a 2-term resolution

Reformulate

To prove:

Lemma

For a torsion free abelian group M and the spectrum HM *with trivial* $\mathbb{T}$*-action, the canonical map*

$$\mathrm{HM}^{\mathrm{t}\mathbb{T}} \to \left(\mathrm{HM}^{\mathrm{tC}_p}\right)^{\mathrm{h}\,\mathbb{T}/\mathrm{C}_p}$$

is a p-completion.

Reduction again

Convergence of the Tate spectral sequences of $\mathrm{HM}^{\mathrm{t}\mathbb{T}}$ and $\mathrm{HM}^{\mathrm{tC}_p}$ leads to:

$$\mathrm{HM}^{\mathrm{t}\mathbb{T}} \simeq \bigoplus_{i\in\mathbb{Z}} \Sigma^{2i}\,\mathrm{HM} \quad \text{and} \quad \mathrm{HM}^{\mathrm{tC}_p} \simeq \bigoplus_{i\in\mathbb{Z}} \Sigma^{2i}\,\mathrm{H}\,\mathrm{M}/\,\mathrm{p}$$

In other words, $\mathrm{HM}^{\mathrm{t}\mathbb{T}}$ is a periodised $\mathrm{HM}^{\mathrm{h}\mathbb{T}}$.
Thus, it suffices to prove

Lemma

The induced map on homotopy groups on each negative even degrees, by the natural map $\mathrm{HM}^{\mathrm{h}\mathbb{T}} \simeq \left(\mathrm{HM}^{\mathrm{hC}_p}\right)^{\mathrm{h}\mathbb{T}} \to \left(\mathrm{HM}^{\mathrm{tC}_p}\right)^{\mathrm{h}\mathbb{T}}$ is the p-completion

$$M \to M_p^{\wedge}.$$

HFPSS

To prove the lemma, we compare the HFPSS's of $\left(\mathrm{HM}^{\mathrm{hC}_p}\right)^{\mathrm{h}\mathbb{T}}$ and $\left(\mathrm{HM}^{\mathrm{tC}_p}\right)^{\mathrm{h}\mathbb{T}}$, respectively:

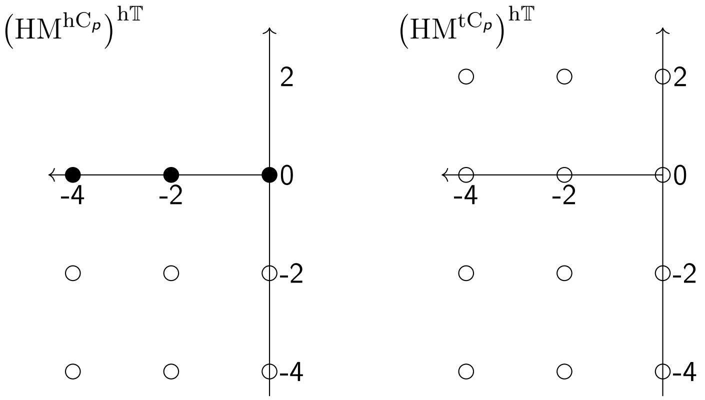
$$E^2_{p,q} = \mathsf{H}^{-p}(\mathrm{B}\mathbb{T}; \pi_q(\mathrm{HM}^{\mathrm{hC}_p})) \Rightarrow \pi_{p+q}\left(\left(\mathrm{HM}^{\mathrm{hC}_p}\right)^{\mathrm{h}\mathbb{T}}\right)$$
$$E^2_{p,q} = \mathsf{H}^{-p}(\mathrm{B}\mathbb{T}; \pi_q(\mathrm{HM}^{\mathrm{tC}_p})) \Rightarrow \pi_{p+q}\left(\left(\mathrm{HM}^{\mathrm{tC}_p}\right)^{\mathrm{h}\mathbb{T}}\right)$$

Remark

Note that we obtain the HFPSS for $X^{\mathrm{h}G} \simeq \mathsf{F}(EG, X)^G$ by either filtering $\mathrm{E}G$ by a suitable filtration, or we consider the tower of fibration induced by the Postnikov tower of X.

HFPSS

In the following picture, we set $\bullet = M$ and $\circ = M/p$



Note that both spectral sequences collapse at E_2-page.

HFPSS of $\left(\mathrm{HM}^{\mathrm{hC}_p}\right)^{\mathrm{h}\mathbb{T}}$

- The spectral sequence computes $\pi_{2k}\left((\mathrm{HM})^{\mathrm{h}\mathbb{T}}\right) \simeq M$
- From the associated graded, we know $\pi_{2k}\left(\left(\mathrm{HM}^{\mathrm{hC}_p}\right)^{\mathrm{h}\mathbb{T}}\right)$ is endowed with the filtration

$$p^k M \subseteq p^{k-1} M \subseteq \cdots \subseteq pM \subseteq M$$

- Truncate the spectral sequence in $x < -2n$ area, we obtain a spectral sequence computing

$$\pi_{2i}\left(\mathsf{F}\left(\mathrm{S}^{2n+1}, \mathrm{HM}^{\mathrm{hC}_p}\right)^{\mathbb{T}}\right) \cong \begin{cases} M, & \text{for } -n \leq i < 0 \\ M/\,p^{n+1} & \text{for } i < -n \end{cases}$$

HFPSS of $\left(\mathrm{HM}^{\mathrm{tC}_p}\right)^{\mathrm{h}\mathbb{T}}$

- The spectral sequence is “periodic”, because $\mathrm{HM}^{\mathrm{tC}_p}$ is.
- Truncate the spectral sequence in $x < -2n$ area and compare it with the truncated HFPSS of $\left(\mathrm{HM}^{\mathrm{hC}_p}\right)^{\mathrm{h}\mathbb{T}}$, we have

$$\pi_{2i}\left(\mathsf{F}\left(\mathrm{S}^{2n+1}, \mathrm{HM}^{\mathrm{tC}_p}\right)^{\mathbb{T}}\right) \simeq M/\,p^{n+1},\ \text{for all } i \in \mathbb{Z},$$

- The filtration $\left\{\pi_{2i}\left(\mathsf{F}\left(\mathrm{S}^{2n+1}, \mathrm{HM}^{\mathrm{tC}_p}\right)^{\mathbb{T}}\right)\right\}_{n\geq 0}$ of $\pi_i\left(\left(\mathrm{HM}^{\mathrm{tC}_p}\right)^{\mathrm{h}\mathbb{T}}\right)$ is complete (?). Thus

$$\pi_i\left(\left(\mathrm{HM}^{\mathrm{tC}_p}\right)^{\mathrm{h}\mathbb{T}}\right) \simeq M_p^{\wedge}$$

End of proof

- The natural map $\mathrm{HM}^{\mathrm{h}\mathbb{T}} \simeq \left(\mathrm{HM}^{\mathrm{hC}_p}\right)^{\mathrm{h}\mathbb{T}} \to \left(\mathrm{HM}^{\mathrm{tC}_p}\right)^{\mathrm{h}\mathbb{T}}$ induces the p-completion map $M \to M_p^\wedge$ on each negative even degree homotopy groups.
- The canonical map

$$\mathrm{HM}^{\mathrm{t}\mathbb{T}} \to \left(\mathrm{HM}^{\mathrm{tC}_p}\right)^{\mathrm{h}\,\mathbb{T}/\mathrm{C}_p}$$

  is the p-completion $M \to M_p^\wedge$, because $\mathrm{HM}^{\mathrm{t}\mathbb{T}}$ is periodised $\mathrm{HM}^{\mathrm{h}\mathbb{T}}$.
- Proof of [NS18, Lemma II.4.2] is complete by the previous reductions.

Summary

- For $X \in \mathbf{Sp}$, we introduce the Tate diagonal

$$\Delta_p(X)\colon X \to \left(X^{\otimes_{\mathbb{S}} p}\right)^{\mathrm{tC}_p}.$$

- We defined the Frobenius $\varphi\colon \mathrm{THH}(R) \to \mathrm{THH}(R)^{\mathrm{tC}_p}$, for $R \in \mathbf{Alg}_{\mathrm{E}_1}(\mathbf{Sp})$.
- We showed that for a bounded below spectrum X with $\mathbb{T}$-action, the canonical map

$$X^{\mathrm{t}\mathbb{T}} \to \left(X^{\mathrm{tC}_p}\right)^{\mathrm{h}\,\mathbb{T}/\mathrm{C}_p}$$

is a p-completion.

References

Achim Krause and Thomas Nikolaus.
Lectures on topological hochschild homology and cyclotomic spectra.

J. McClure, R. Schwänzl, and R. Vogt.
$THH(R) \cong R \otimes S^1$ for E_∞ ring spectra.
J. Pure Appl. Algebra, 121(2):137–159, 1997.

Thomas Nikolaus and Peter Scholze.
On topological cyclic homology.
Acta Math., 221(2):203–409, 2018.