The Alexander Polynomial

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Zusammenfassung

Das Ziel dieser Bachelorarbeit ist die Berechnung der Alexander-Polynome von Torusknoten und Twist-Knoten. Hier ist eine kurze Zusammenfassung des Inhalts dieser Arbeit. In Kapitel 1 führen wir fundamentale Definitionen und Beispiele ein. Mit der Notation von Kapitel 1, stellen wir in Kapitel 2 eine wichtige Knoteninvariante von Kodimension 2 vor, die Knotengruppe. Ab Kapitel 3 betrachten wir klassische Knoten. Zuerst lernen wir die Seifertfläche M eines Knoten K kennen, die eine kompakte orientierbare Fläche mit $\partial M = K$ ist. Mit der Seifertfläche konstruieren wir eine universelle abelsche Überlagerung $\tilde{X}$ des Knotenkomplements $X = S^3 - K$, damit in Kapitel 4 $H_1(\tilde{X})$ berechnet werden kann. In Kapitel 4.3 definieren wir das Alexander-Polynom eines Knotens als die Determinante einer Seifert-Matrix, die aus einer beliebigen Seifertfläche des Knotens berechnet werden kann. Am Ende der Arbeit dient die Berechnung des Alexander-Polynoms des Twist-Knotens als eine Anwendung dieser Definition.

Introduction

A knot K in a space X is a subspace which is homeomorphic to a sphere. In this thesis, it is assumed most of the time that $K \approx S^1$ and $X = \mathbb{R}^3$ or S^3, which is classical knot theory. In this thesis, we call two knots equivalent if and only if they are ambiently homeomorphic (Definition 1.3). For example, there are only two equivalence classes of knots in the 2-torus. But there are infinitely many equivalence classes of torus knots, which are the knots embedded in the standard torus as a subspace of S^3. In the first case, the whole space X is the torus, where as in the latter one has $X = S^3$. In order to classify knots, we need knot invariants which assign to equivalent knots the same object. For a one-dimensional knot K in S^3, many knot invariants have been constructed. For instance, the fundamental group of a knot complement (the knot group) is a useful knot invariant in codimension 2. Also from a Seifert surface (Definition 3.2) of a knot we can compute the torsion invariant [Rol76, Chapter 6] and the Alexander invariant (Chapter 4.1). Besides there are more recent development of knot invariants like Jones polynomials and Khovanov homology.

The purpose of this Bachelor thesis is computing the Alexander polynomials $\Delta(t)$ (Definition 4.15) of torus knots (Definition 2.7) and twist knots (Chapter 1.2). Torus knots are defined as the images of embeddings from S^1 into the standard torus, considered as a subspace in S^3. For example, the trefoil knot is a torus knot.

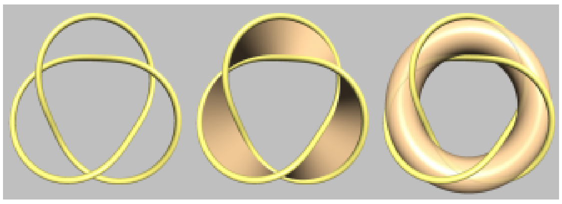

FIGURE 1: Trefoil knot embedded on a torus [WC06]

As for the twist knot, consider a handle-body decomposition of a compact, connected genus one surface M with boundary:

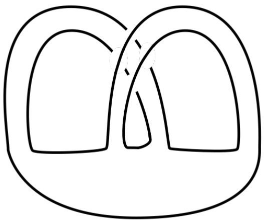

FIGURE 2: A compact connected genus 1 surface with boundary

Define the twist knot $K_{m,n}$ as the boundary of M after fully twisting each handle m and n times respectively with $m,\ n$ integers:

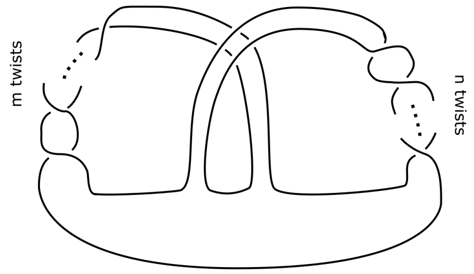

FIGURE 3: $K_{m,n}$ with $m < 0$ and $n < 0$

For a one-dimensional knot K in S^3, the Alexander polynomial is a knot invariant that describes the Λ-module structure of the first homology group $H_1(\tilde{X})$ of the universal abelian cover $\tilde{X}$ of the knot complement, where Λ is the ring of Laurent polynomials over $\mathbb{Z}$. We introduce first two explicit methods to compute a Λ-module presentation of $H_1(\tilde{X})$:

(a) Computation using the knot group (Chapter 4.1.2).

(b) Computation using a Seifert surface S to construct the universal abelian cover $\tilde{X}$(Chapter 4.1.1);

(c) Computation using the Seifert matrix of S

Using (b), we compute the Alexander polynomial of a torus knot $T_{p,q}$ (Chapter 4.1.3) :

$$\Delta(t)_{p,q} = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)}.$$

Using (c), we compute the Alexander polynomial of a torus knot $T_{p,q}$ (Chapter 4.3.2) :

$$\Delta(t)_{m,n} = mnt^2 + (1-2mn)t + mn.$$

As consequences we get

(1) For $p,\ q > 1$, two torus knots T_{p_1,q_1}, T_{p_2,q_2} are equivalent if and only if $\{p_1, q_2\} = \{p_2, q_2\}$;

(2) For $mn \neq 0$, the product mn is an invariant of the twist knot $K_{m,n}$. We need additional knot invariants to distinguish $K_{m,n}$ and $K_{m',n'}$ with $mn = m'n'$;

(3) $K_{1,1}$ and $K_{-1,-1}$ are the only twist knots that can be embedded in a torus, i.e. they are also torus knots. In particular $K_{1,1}$ is a right-handed trefoil knot and $K_{-1,-1}$ is a left-handed trefoil knot.

Here is a brief description of the structure of this thesis. Chapter 1 is devoted to introducing the basic definitions of knots and knot equivalence and providing the readers with some examples of classical knots in S^3. In chapter 2, we present an important knot invariant in codimension 2, the knot group. From Chapter 3 we restrict our attention to classical knots. First we introduce Seifert surfaces of a knot which are compact, connected oriented surfaces whose boundaries are the knot. Then we construct a universal abelian cover of the knot complement using a Seifert surface of the knot, which will be used at Chapter 4.1 to give two methods for computing the first homology group of the universal abelian cover $H_1(\tilde{X})$. In Chapter 4.3, the Alexander polynomial of a knot $K \subset S^3$ is defined as the determinant of a Seifert matrix, a matrix with polynomial entries computed from an arbitrary Seifert surface of K. To close, we compute the Alexander polynomials of twist knots from this definition.

The main reference for this thesis is [Rol76]. In addition, we refer to [Mas77] for an explanation of covering spaces and properties of free groups.

Acknowledgements

First of all, I would like to express my sincere thanks to my advisor Prof. Dr. Peter Teichner for his support and patience. During our discussions of this topic he always gave me insightful advices on the problems that I have while writing this thesis and raised new questions for me to think about, which was very inspiring. I also want to thank Achim Krause, Bertram Arnold and Jonathan Silver and Jasper Knyphausen for proofreading my manuscript in detail, correcting linguistic errors and most important of all, for the discussions that we had on many aspects of this thesis.

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Besides, I wish to express my gratitude to my parents who always respect my choices and support me and Johanna Kaster for her love and care as if I were her own daughter.

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Contents

Chapter 1

Definitions and Examples

This chapter aims at giving an elementary introduction to knot theory, including basic definitions and some examples of classical knots. The main reference for this chapter is [Rol76, Chapter 1].

1.1 Notation and Definitions

Definition 1.1. A subset K of a space X is a *knot* in X if K is homeomorphic to a sphere S^p.

Unless specifically mentioned, all knots are of codimension 2, i.e. we take $p = n$, $X \approx \mathbb{R}^{n+2}$ or $X \approx S^{n+2}$.

Remark 1.2. In our definition we do not take the orientation of the knot into consideration. There are other different definitions of knots besides the one above. For example, in some sources a knot is defined as an embedding $K : S^p \hookrightarrow X$ instead of a subset of X in which orientations may play a role.

We can also define different equivalence relations for knots as follows:

Definition 1.3. Given two knots K, K', these are

- *ambiently homeomorphic* if there is a self-homeomorphism $h : X \longrightarrow X$ such that $h(K) = K'$. In other words $(X, K) \approx (X, K')$;
- *orientation-preservingly homeomorphic* if there is an ambient homeomorphism $h : (X, K) \longrightarrow (X, K)$ and h is orientation-preserving;

- *ambiently isotopic* if there is an ambient homeomorphism $h : (X, K) \longrightarrow (X, K)$ that is ambiently isotopic to $id : X \longrightarrow X$.

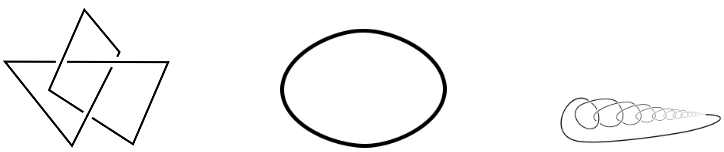
Knots in an equivalence class are called knots of the same *knot type*.

Remark 1.4. According to the Alexander-Guggenheim theorem, orientation-preserving homeomorphism (isoposition) is equivalent to piecewise linear ambient isotopy. For a proof, see [RS72, Chapter 3, 4].

In this thesis, we say for brevity two knots are equivalent if they are ambiently homeomorphic. If one of the other two equivalence relations is used, we will mention it explicitly.

1.2 Examples

1. A *polygonal knot* in $\mathbb{R}^3$ is a knot which is a union of finite line segments. A *tame knot* is a knot which is equivalent to a polygonal knot. Knots which are not tame are *wild*.



(A) Polygonal Knot (B) Unknot (C) Wild knot

2. The subspace $S^{n-2} \subseteq \mathbb{R}^{n-1} \subseteq \mathbb{R}^n \subseteq \mathbb{R}^n + \infty \approx S^n$ in S^n is called the *trivial knot* or *unknot* in codimension 2.

3. Define the *standard torus* T in S^3 as the image of the embedding i_T:

$$\begin{array}{rcl} S^1 \times S^1 & \hookrightarrow & R^3 \subset S^3 \\ (\cos\theta, \sin\theta, \cos\varphi, \sin\varphi) & \mapsto & (\cos\theta + \cos\varphi\cos\theta, \sin\theta + \cos\varphi\sin\theta, 1 + \sin\varphi)\ . \end{array}$$

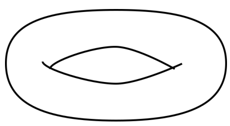


FIGURE 1.2: Standard torus

The *trefoil knot* $T_{2,3}$ in $\mathbb{R}^3$ or S^3 is defined as the image of the embedding in S^3:

$$\begin{aligned} S^1 &\to T \hookrightarrow S^3 \\ z &\mapsto (z^2, z^3) \mapsto i_T(z^2, z^3) \ . \end{aligned}$$

The trefoil knot belongs to the family of *torus knots* (Definition 2.7) and *twist knots* (below). Also there are the *left-handed trefoil* and *right-handed trefoil*, which are ambiently homeomorphic to each other by reflection. However the reflection is not orientation-preserving. In 1914 Max Dehn showed that left-handed trefoil knot and right-handed trefoil knot are not ambiently isotopic to each other [Deh14].

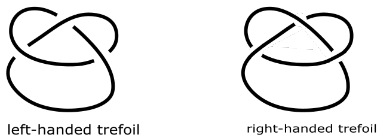


FIGURE 1.3: Left-hand and right-hand trefoils

4. Consider the handlebody decomposition of the compact connected oriented surface M of genus 1 with boundary which has exactly two 1-handles. The *twist knot* $K_{m,n}$ is defined as the boundary of M after fully twisting the two handles m and n times respectively where m and n are integers. Define the left diagram below as +1 full twist and the right diagram below as -1 full twist.

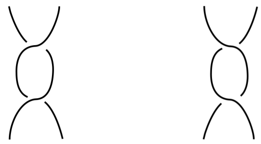


FIGURE 1.4: Positive (left) and negative (right) full twists

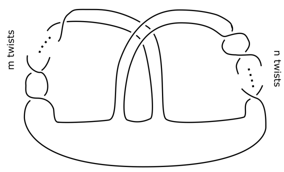


FIGURE 1.5: An Example of $K_{mn}$with $m < 0, \ n < 0$

Remark 1.5.

- It doesn't matter which handle is in front in the projection, since by an isotopy these two knots can be transformed into each other, i.e. defines equivalent knots. We show this by pictures:

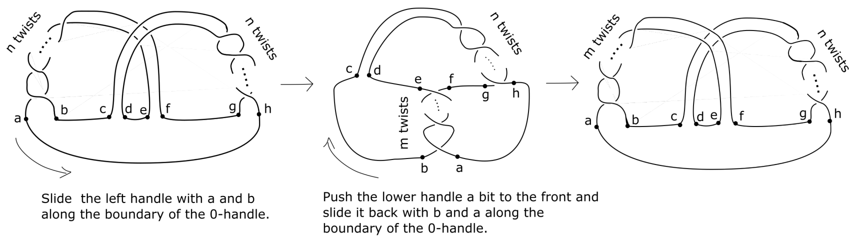


FIGURE 1.6

- $K_{m,n}$ and $K_{n,m}$ define the same knot since this depends merely on to which plane we project the knots.
- $K_{m,n}$ and $K_{-n,-m}$ are equivalent knot since they differ by a reflection.

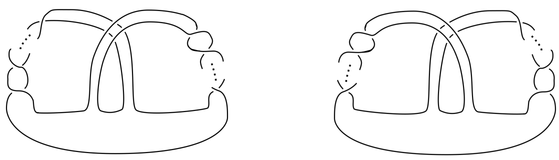


FIGURE 1.7

Chapter 2

Knot Invariants and Torus knots

Just like topological spaces, knots can be classified using knot invariants.

Definition 2.1. A *knot invariant* is a function $K \mapsto f(k)$ which assigns to each knot K an object $f(K)$ in such a way that knots of the same type are assigned equivalent objects.

This chapter introduces an important and useful knot invariant in codimension 2, the knot group, and its application to torus knots. The main reference for this chapter is [Rol76, Chapter 2].

Remark 2.2. One hopes that $f(K)$ is on the one hand easy to calculate, and on the other hand, sensitive enough to solve the problem at hand. In the case of codimension 2, the fundamental group of the knot complement is a very useful tool. In chapter 4 we shall introduce two more invariants: the Alexander invariant and the Alexander polynomial.

2.1 The knot group

Theorem 2.3 (Gordon-Luecke, 1989)**.** *Two tame knots in $\mathbb{R}^3$ or S^3 are equivalent if and only if they have homeomorphic knot complements.*

For a proof of this theorem, see [GL89].

Most of the time it is hard to describe or characterize the complement of a knot topologically. However by composing the function c with any functor F from topological spaces to an algebraic category will give a knot invariant, sending K to $F(X - K)$.

In codimension 2, the knot group is the an important knot invariant.

Definition 2.4. If K is a knot in $\mathbb{R}^n$, the fundamental group $\pi_1(\mathbb{R}^n - K)$ of its complement in $\mathbb{R}^n$ is called the *knot group* of K.

Using the proposition below we can also define the knot group as $\pi_1(S^3 - K)$.

Proposition 2.5. *If B is a bounded subset of $\mathbb{R}^n$ such that $\mathbb{R}^n - B$ is path-connected and $n \geq 3$, then the natural inclusion $i : \mathbb{R}^n - B \longrightarrow S^n - B$ induces an isomorphism:*

$$i_* : \pi_1(\mathbb{R}^n - B) \longrightarrow \pi_1(S^n - B) \ .$$

Proof. Choose any neighbourhood U of ∞ in S^n which misses B and is itself homeomorphic to $\mathbb{R}^n$. Then $S^n - B \approx (\mathbb{R}^n - B) \cup U$, $(\mathbb{R}^n - B) \cap U \approx U - \infty \approx S^{n-1}$. According to the Seifert-van Kampen theorem, the following commutative diagram is a push out diagram in groups:

$$\begin{array}{ccc} \pi_1(S^{n-1}) & \xrightarrow{i_*} & \pi_1(\mathbb{R}^n - B) \\ \downarrow{i_*} & & \downarrow{i_*} \\ \pi_1(U) & \xrightarrow{i_*} & \pi_1(S^n - B) \ . \end{array}$$

Since S^{n-1} and U are simply connected,

$$i_* : \pi_1(\mathbb{R}^n - B) \to \pi_1(S^n - B)$$

is an group isomorphism. □

Remark 2.6. For tame knots, knot groups are only interesting in codimension two: by transversality theorem [W.H33, Theorem 2.1], a knot $K \approx S^k$ in S^n has simply-connected complement if $n - k \geq 3$.

Now we are going to take a look at an important family of nontrivial knots in S^3, the torus knots.

2.2 Torus knots

Definition 2.7. The image of any embedding $t : S^1 \to T = S^1 \times S^1 \subset S^3$ is a *torus knot*, where T denotes the standard torus in S^3.

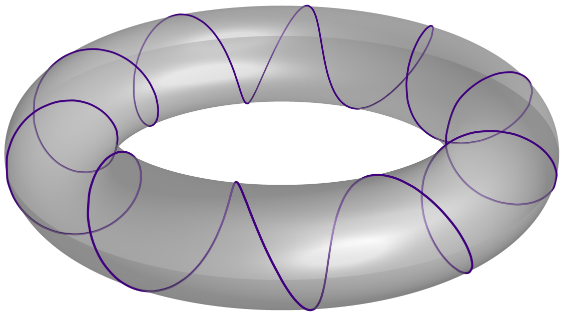


FIGURE 2.1: A torus knot

Theorem 2.8. *Any embedding* $t : S^1 \to S^1 \times S^1$ *is ambiently isotopic to an embedding* $t_{p,q}$ *of the form:*

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^1 \times S^1 \\ z & \mapsto & (z^p, z^q) \end{array}$$

with $|p|, |q|$ *coprime non-negative integers.*

Proof. Each embedding $i : S^1 \to T$ determines an equivalence class of $H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$, which we shall denote by $[i]$. Furthermore $[i] = (a, b)$ with $\gcd(a, b) = 1$ [Rol76, Theorem 2.C.2].

From [Rol76, Theorem 2.C.16] we conclude that the images J, K of the two embeddings $j, k : S^1 \to T$ respectively are ambiently isotopic as subspaces of T if and only if $[j] = \pm [k]$. By isotopy extension theorem [W.H33, Chapter 8, Theorem 1.3] the ambiently isotopy above can be extended to an ambient isotopy in S^3.

On the other hand, each equivalence class (a, b) of $\pi_1(T)$ with $a, b \in \mathbb{Z}$ is represented by

$$\begin{aligned} S^1 &\to S^1 \times S^1 \\ z &\mapsto (z^a, z^b) \ . \end{aligned}$$

□

Remark 2.9. Composing the map $t_{p,q}$ with projections onto the first and second component respectively, we observe that the image of the embedding $t_{p,q}$ is a knot which wraps around the torus along the longitude $|p|$ times and the meridian $|q|$ times.

Remark 2.10. The type of $T_{p,q}$ does not change by changing the sign of p or q, or by interchanging p and q, since the image does not change when we change the signs.

Thus we only need to study those embeddings $t_{p,q}$ with $p > 0, q > 0$, $\gcd(p, q) = 1$. We denote the image of the embedding $t_{p,q}$ by $T_{p,q}$.

Proposition 2.11. *The knot group of $T_{p,q}$ is $G_{p,q} := \langle x, y \mid x^p = y^q \rangle$.*

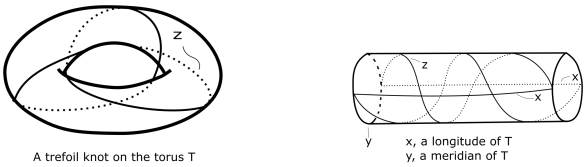


FIGURE 2.2: Trefoil embedded on a torus and the relation of x, y, z

Proof. Consider S^3 as the union of two solid tori T_1, T_2 that have as their common boundary the torus T

$$\begin{aligned} S^3 &= \partial D^4 \\ &= \partial(D^2 \times D^2) \\ &= S^1 \times D^2 \sqcup_{S^1 \times S^1} D^2 \times S^1 \\ &= T_1 \sqcup_T T_2 \ . \end{aligned} \tag{2.1}$$

We have

$$S^3 - T_{p,q} = (T_1 - T_{p,q}) \sqcup_{T - T_{p,q}} (T_2 - T_{p,q})$$

$$(T_1 - T_{p,q}) \cap (T_2 - T_{p,q}) = T - T_{p,q} \ .$$

Let the torus knot $T_{p,q}$ be embedded on T via the embedding $t_{p,q}$. We shall denote $z := t_{p,q}$, x a representative of $\big[(1,0)\big] \in \pi_1(T)$ and y a representative of $\big[(0,1)\big] \in \pi_1(T)$. Furthermore we have

$$T_1 - T_{p,q} \simeq T_1 \simeq S^1 \text{ via } x$$

$$T_2 - T_{p,q} \simeq T_2 \simeq S^1 \text{ via } y \ .$$

Therefore we obtain

$$\pi_1(T_1 - T_{p,q}) = \langle x \rangle$$

$$\pi_1(T_2 - T_{p,q}) = \langle y \rangle$$

$$\pi_1(T - T_{p,q}) = \langle z \rangle \ .$$

From remark 2.9 we conclude that

$$z = x^p$$

$$z = y^q \ .$$

By Seifert-van Kampen theorem:

$$G_{p,q} = \pi_1(S^3 - T_{p,q}) \;=\; \langle x, y \mid x^p = y^q \rangle \;.$$

□

Theorem 2.12 (O.Schreier). *If $p, q > 1$, then the group $G_{p,q}$ determines the pair $\{p, q\}$.*

A proof of this theorem using the Alexander polynomial will be given in Chapter 4. The original proof given by Schreier can be found in [Sch24].

An immediate result of this theorem is:

Corollary 2.13. *There exist infinitely many torus knot types.*

Chapter 3

Seifert surfaces

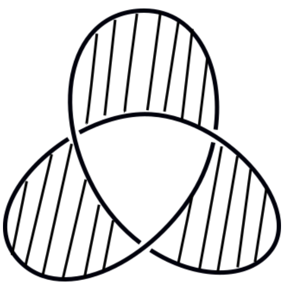
One way to compute the Alexander polynomial of a given knot is by using a Seifert surface. In this chapter, we define Seifert surfaces and derive some of their properties. In the last section we construct an infinite cyclic covering of a knot complement by using a Seifert surface of the knot. This infinite cyclic covering leads to the computation of the Alexander invariant and the Alexander polynomial in Chapter 4. The main reference for this chapter is [Rol76, Chapter 5].

3.1 Surfaces and genus

Definition 3.1. A subset $X \subset Y$ is said to be *bicollared* in Y if there exists an embedding $b : X \times [-1, 1] \to Y$ such that $b(x, 0) = x$. The map b or its image is then said to be the *bicollar* of X.

Definition 3.2. A *Seifert surface* of a knot $K^n \subset S^{n+2}$ is a connected, bicollared (as a subset of S^3), compact manifold $M^{n+1} \subset S^{n+2}$ with $\partial M = K$.

Remark 3.3. From the definition Seifert surfaces are oriented, otherwise they are not bicollared.



(A) The möbius strip with a trefoil knot as its boundary (unorientable)

(B) A Seifert surface of a trefoil knot

In the classical case ($n = 1$), the Seifert surface is a compact connected oriented surface which can be completely classified by its genus.

Theorem 3.4 (Classification of surface)**.** *Every closed orientable connected surface is homeomorphic to one which appears in the table below, and is classified by its genus $g \geq 0$. Two compact connected surface M, M' with boundary are homeomorphic if and only if they have the same number of boundary components and their associated closed surfaces $M \sqcup_{\partial M} (\sqcup_{i=1}^{n} D^2)$ and $M' \sqcup_{\partial M} (\sqcup_{i=1}^{n} D^2)$ are homeomorphic. Here n denotes the number of boundary components of M respectively M'.*

Manifold	S^2	T^2	$T^2\#T^2$	$T^2\#...\#T^2$
Genus	*0*	*1*	*2*	g = *number of* T^2
Euler characteristic	*2*	*0*	*-2*	*2-2g*

Definition 3.5. The genus of a compact connected oriented surface M with boundary is defined to be the genus of its associated closed surface $\hat{M} = M \sqcup_{\partial M} (\sqcup_{i=1}^{n} D^2)$ where n is the number of its boundary components.

We can now make the following statement:

Proposition 3.6. *Let M be a compact connected oriented surface. Then $g(M) = 1 - \frac{\chi(M)+b}{2}$ where b is the number of boundary components.*

Definition 3.7. The *genus g(K) of a knot $K \approx S^1$* in $\mathbb{R}^3$ or S^3 is the least genus of all its Seifert surfaces.

Remark 3.8. If two knots K_1, K_2 in S^3 are equivalent, the ambient homeomorphism $h : (S^3, K) \to (S^3, K)$ gives a homeomorphism between Seifert surfaces of K_1 and K_2 preserving the genus. Together with the following existence theorem the genus of a knot is a knot invariant.

Theorem 3.9 (Existence theorem)**.** *Every knot $K \approx S^1$ in $\mathbb{R}^3$ or S^3 has a Seifert surface.*

Proof. Let K denote the knot in $\mathbb{R}^3$ or S^3 and assign K an orientation and examine a regular projection [HH63, Chapter I, page 6-8]. Near each crossing point, delete the over- and undercrossings and replace them by "shortcut" arcs in the projection plane as pictured, so that the orientation of the line segments is preserved.

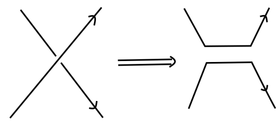


FIGURE 3.2: Introducing a “shortcut” arc

We now have a collection of disjoint simple closed oriented curves in the plane which we shall call *Seifert circles*. Each Seifert circle bounds a disk in the plane, and although they may be nested, these disks can be made disjoint by pushing their interiors slightly off the plane, starting with the innermost ones and working outward. Moreover these disks have bicolloars which may be locally assigned a “ + ” and “ − ” side according to the convention, say, that the oriented boundary runs counterclockwise as seen from the + side.

We now connect these disks together at the old crossings with half twisted strips to form an oriented surface M whose boundary is the original knot K.

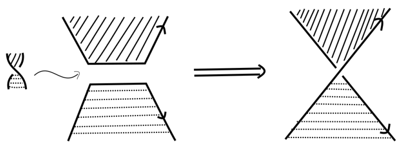


FIGURE 3.3: Connecting the disks

□

Remark 3.10. The above explicit construction is called the *Seifert algorithm*. There are also existence theorems for higher dimensional Seifert surfaces, see [Rol76, Chapter 5.B].

A useful proposition follows:

Proposition 3.11. *Given a regular projection of a tame knot K and construct a Seifert surface M of K using the Seifert algorithm, let c be the number of crossings and s be the number of Seifert circles. Then the Seifert surface constructed has genus*

$$g(M) = 1 - \frac{s+1-c}{2} \ .$$

Proof. We can deformation retract Seifert circles to points and crossings connecting Seifert circles to curves connecting the points. Then the Seifert surface has the homotopy type of a CW-complex with s 0-cells and c 1-cells. Thus

$$\chi(M) = s - c\ .$$

Therefore $g(M) = 1 - \frac{(s-c)+1}{2} = 1 - \frac{s+1-c}{2}$. □

Proposition 3.12. *A knot $K \approx S^1$ in S^3 has genus zero if and only if it is a unknot.*

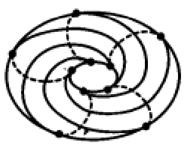
Proof. If K is an unknot, K bounds a disk in S^3, which has genus zero. Thus $g(K) = 0$.

If $g(K) = 0$, K is the boundary of a Seifert surface with genus 0. A genus 0 surface with one boundary component is a disk. Thus K is the boundary of a disk, i.e. an unknot. □

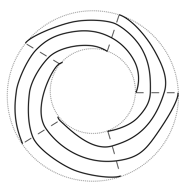
Proposition 3.13. *The torus knot of type $T_{p,q}$ has genus $g(T_{p,q}) \leq \frac{(p-1)(q-1)}{2}$.*

Proof. For $T_{p,q}$, we can construct a representative of $T_{p,q}$ on a standard torus [Ada04, Chapter 5, 109-110] with a regular projection constructed by the following steps:

1. Mark the following $2q$ points in $\mathbb{R}^2$ (in polar coordinate): $\{(1,k\phi)\}_{k=1,2,\ldots,q}$ and $\{(2,k\phi)\}_{k=1,2,\ldots,q}$, $\phi = \frac{2\pi}{q}$;
2. Connect $(1,k\phi)$ with $(2,k\phi)$ by q line segments, $k = 1,2,\ldots,q$;
3. Construct p arcs $p_n = \{(r,\theta) | \frac{\theta-\phi}{r-(2-\frac{n}{p})} = \frac{\phi}{\frac{1}{p}} = p\phi\}$ (i.e. these arcs are linear with respect to polar coordinates), where the nth arc has endpoints $(2-\frac{n}{p},\phi)$ and $(2-\frac{n+1}{p},0)$, $n = 0,1,\ldots,(p-1)$. Note that no two of these curves intersect each other;
4. Construct arcs connecting $(2-\frac{n}{p},k\phi)$ and $(2-\frac{n+1}{p},(k-1)\phi)$, $n = 0,1,\ldots,(p-1)$, $k = 2,3,\ldots,q$ by rotating the arcs constructed in Step 3 counterclockwise $(k-1)\phi$ around the origin. Then we obtain p pairwise non-intersecting strands consisting of those arcs and the arcs constructed in Step 3. Let these p strands cross over the q line segments constructed in Step 2.



(A) T_{35} [Ada04, Chapter 5, Figure 5.9]



(B) A regular projection of T_{35}

Applying the Seifert algorithm to this projection we create "shortcut" arcs connecting each pair of adjacent images of p_i under the rotations in Step 3. Therefore we get p Seifert circles $\{s_i\}_{i=0,1,\dots,p-1}$. Here s_i consists of the arc p_i, its images under rotation in Step 4 and the short cuts connecting two adjacent arcs among them. Furthermore we have $q(p-1)$ crossings in total.

FIGURE 3.5: The crossings(left) and after applying the Seifert algorithm(right)

Using the formula in the previous proposition:

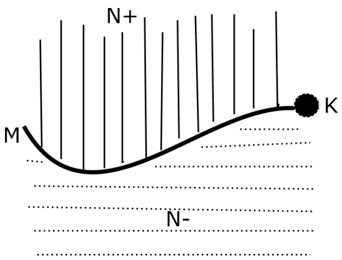
$$g(M_{p,q}) = 1 - \frac{p-(p-1)q+1}{2} = \frac{(p-1)(q-1)}{2} .$$

□

3.2 The cyclic covering of a knot complement

As mentioned at the beginning of this chapter there is an important class of covering spaces of a knot complement $X = S^{n+2} - K$, $K \approx S^n$, which will be used in the next chapter to define the Alexander invariant and the Alexander polynomial of K. A Seifert surface of K allows us to construct these covering spaces explicitly.

Let M be a Seifert surface for the knot $K \approx S^n$ in S^{n+2} and let $\varphi : \mathring{M} \times (-1,1) \hookrightarrow S^{n+2}$ be an embedding such that $\mathring{M} = \varphi(\mathring{M} \times 0)$, i.e. φ is a bicollar of the interior of M. We denote:



$$N = \varphi(\mathring{M} \times (-1,1))$$

$$N^+ = \varphi(\mathring{M} \times (0,1))$$

$$N^- = \varphi(\mathring{M} \times (-1,0))$$

$$Y = S^{n+2} - M$$

$$X = S^{n+2} - K$$

In this way we have two triples (N, N^+, N^-) and (Y, N^+, N^-). Let us form countably many copies of each, denoted (N_i, N_i^+, N_i^-) and (Y_i, N_i^+, N_i^-), $i \in \mathbb{Z}$. Let $\tilde{N} = \sqcup_{i\in\mathbb{Z}} N_i$ and $\tilde{Y} = \sqcup_{i\in\mathbb{Z}} Y_i$ be the disjont unions. Finally, we form an identification space $\tilde{X}$ by identifying $N_i^+ \subset Y_i$ with $N_i^+ \subset N_i$ via the identity homeomorphism, and likewise identify each $N_i^- \subset Y_i$ with $N_{i+1}^- \subset N_i$. We shall call the resulting space $\tilde{X}$.

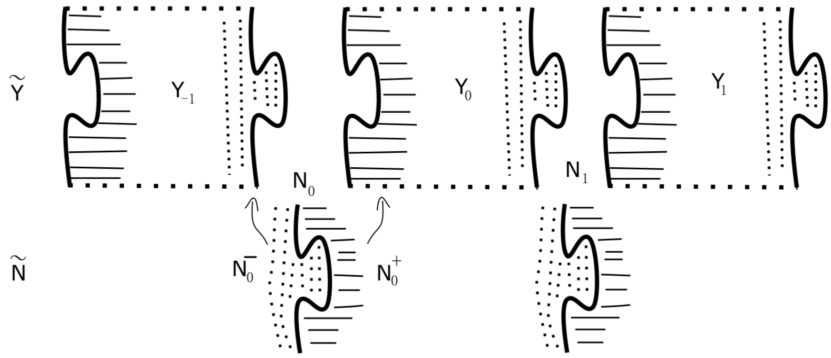


FIGURE 3.6: Construction of X

Proposition 3.14. *$\tilde{X}$ is a path-connected non-compact $(n+2)$-manifold. There is map $p : \tilde{X} \to X$ which is a regular covering space. There is a covering automorphism $\tau : \tilde{X} \to \tilde{X}$, which takes Y_i to Y_{i+1} and N_i to N_{i+1}, and τ generates the deck transformation* $\mathrm{Aut}_X \tilde{X}$.

Proof.

- **$\tilde{X}$ is path connected**. We know that each Y_i, N_i is path-connected, $i \in \mathbb{Z}$. Pick arbitrary points $y_i \in Y_i$, $y_{i+1} \in Y_{i+1}$, there are points $n_{i+1}^- \in N_{i+1}^- \subset Y_i$ and $n_{i+1}^+ \in N_{i+1}^+ \subset Y_{i+1}$ such that y_i and n_{i+1}^-, n_{i+1}^- and n_{i+1}^+, n_{i+1}^+ and y_{i+1} are path connected. Thus y_i and y_{i+1} are path connected.

- **Regular covering**

  Define $p : \tilde{X} \to X$ by

$$p|_{Y_i} : Y_i \xrightarrow{natural\ inclusion} X, \forall i \in \mathbb{N}$$

$$p|_{N_i} : N_j \xrightarrow{natural\ inclusion} X, \forall j \in \mathbb{N} \ .$$

  It follows from the construction that p is well-defined at the intersection of Y_i and N_j. Thus p is well-defined on $\tilde{X}$.

  p is continuous because $\forall U \subset X$ open subspaces:

$$p^{-1}(U) = \cup(U \cap ((Y_{i-1} \dot{\bigcup}_{N_i} Y_i)))$$

which is an infinite union of open sets and therefore is again open.

p is a covering map because:

$\forall x \in Y - N \subset X,\ \exists U_x \subset Y,\ U_x$ an open neighbourhood of x, since Y is open. Thus

$$p^{-1}(U_x) = \cup(U_x \cap Y_i)$$

where for each i, $U_x \cap Y_i$ are disjoint and mapped homeomorphically to U_x. The same applies to the point in N. In particular, this is an infinite covering.

Since $\tilde{X}$ is a covering space of X, a $(n+2)$-manifold, $\tilde{X}$ is also a $(n+2)$-manifold. Furthermore X is non-compact and p is surjective, thus $\tilde{X}$ is also non-compact.

Now let's consider the deck transformations of $\tilde{X}$.

Define a $\mathbb{Z}$ action on $\tilde{X}$ by

$$\begin{array}{rcl} \mathbb{Z} \times \tilde{X} & \to & \tilde{X} \\ (k, y_i) & \mapsto & k.y_i = y_{i+k} \\ (k, n_i) & \mapsto & k.n_i = n_{i+k} \end{array}$$

where $y_i \in Y_i,\ y_{i+k} \in Y_{i+k},\ n_i \in N_i,\ n_{i+k} \in N_{i+k},\ p(y_i) = p(y_{i+k}), p(n_i) = p(n_{i+k}),\ i \in \mathbb{Z},\ k \in \mathbb{Z}$.

This action is free since if $ky_i = y_i,\ y_i \in Y_i, i \in \mathbb{Z}$, then $k = 0$, the same works for points in $N_i, i \in \mathbb{Z}$.

This action is transitive on the fibers since for all $y \in Y \subset X$, $p^{-1}(y) = \{y_i\}_{i\in\mathbb{Z}}$ with $y_i \in Y_i$ and $(j-i, y_i) = y_j, \forall i \in \mathbb{Z},\ , j \in \mathbb{Z}$. The same works for points in $N_i, i \in \mathbb{Z}$.

Therefore p is a regular covering map and $\mathrm{Aut}_X \tilde{X} = \mathbb{Z}$. In particular, $\tau_+ : \tilde{X} \to \tilde{X}$ defined by $\tau_+(\tilde{x}) = 1.\tilde{x}$ and $\tau_- : \tilde{X} \to \tilde{X}$ defined by $\tau_-(\tilde{x}) = (-1).\tilde{x}$ are generators of $\mathrm{Aut}_X \tilde{X}$.

□

Definition 3.15. $\tilde{X}$ is called an *infinite cyclic cover* of the knot complement X.

Proposition 3.16. *$\tilde{X}$ is the universal abelian cover of X.*

Proof. There is the short exact sequence:

$$1 \to \pi_1(\tilde{X}) \xrightarrow{p_*} \pi_1(X) \to \mathrm{Aut}_X \tilde{X} \cong \mathbb{Z} \to 1\ .$$

Since $\pi_1(\tilde{X})/p_*(\pi_1(X)) \cong \mathbb{Z}$ is abelian, $C = \left[\pi_1(X), \pi_1(X)\right] \subset p_*(\pi_1(\tilde{X}))$.

Thus we have a surjective group homomorphism:

$$\mathbb{Z} \cong \pi_1(X)/C \twoheadrightarrow \pi_1(X)/p_*(\pi_1(\tilde{X})) \cong \mathbb{Z}$$

which is an isomorphism with kernel $p_*(\pi_1(\tilde{X}))/C = 0$. So we have

$$p_*(\pi_1(\tilde{X})) = C\ .$$

Therefore $\tilde{X}$ is the universal abelian cover. □

Corollary 3.17. *$\tilde{X}$ depends (up to covering isomorphism) only on the knot type of K, and not on the choice of Seifert surface or other choices in the above construction.*

Chapter 4

The Alexander Polynomial and the Alexander Invariant

The Alexander invariant is a knot invariant defined as the homology groups of the universal abelian cover $H_*(\tilde{X})$ as modules of the ring of Laurent polynomials. The Alexander polynomial is a description of $H_1(\tilde{X})$. The two families of knots that are studied in this thesis are both classical knots whose higher homology groups $H_k(\tilde{X})(k \geq 2)$ are trivial. Therefore, after a short introduction to Alexander invariants, we will put our main focus on the computation of the Alexander polynomials, some related properties and two applications. The main references for this chapter is [Rol76, Chapter 5, 7 and 8].

4.1 The Alexander invariant

Definition 4.1. Let Λ denote the ring of *Laurent polynomials* with integer coefficients and one variable t. An element of Λ has the form:

$$p(t) = \sum_{i=r}^{s} c_i t^i, c_i \in \mathbb{Z}, \ r, s \in \mathbb{Z}, r \leq s.$$

Addition and multiplication are as usual with polynomials. One can also write Λ as $\mathbb{Z}[t, t^{-1}]$. The units of Λ are the monomials $\pm t^i, i \in \mathbb{Z}$.

Given a knot $K^n \subset S^{n+2}$ with complement X. Let $\tilde{X}$ be the universal abelian covering space of X. Fix one of the two generators $\tau : \tilde{X} \to \tilde{X}$ of the group of deck transformations and define the Λ-module structure of $H_*(\tilde{X})$ by:

$$\sum_{i=r}^{s} c_i t^i(\alpha) = \sum_{i=r}^{s} c_i \tau_*^i \alpha$$

with $\alpha \in H_i(\tilde{X})$. $\tau_* : H_i(\tilde{X}) \to H_i(\tilde{X})$ is the homology isomorphism induced by τ. Thus $\sum_{i=r}^{s} c_i t^i(\alpha)$ is again an element of $H_i(\tilde{X})$.

Theorem 4.2. *The above Λ-multiplication gives a Λ-module structure on $H_i\tilde{X}, \forall i \in \mathbb{N}$. Equivalent knots have isomorphic Alexander invariants as Λ-modules in each dimension, modulo appropriate choices of τ.*

Proof. It's easy to see that the Λ-multiplication gives a well-defined module structure on $H_i\tilde{X}, \forall i \in \mathbb{N}$. Now we check that the Alexander invariant is a knot invariant in each dimension.

Given two equivalent knots K_1, K_2 in S^{n+2}, there is an ambient homeomorphism $h : S^{n+2} \to S^{n+2}$ such that $h(K_1) = K_2$. Thus $h(X_1) = X_2$, where X_i is the knot complement of K_i, $i = 1, 2$. Thus we have the following commutative diagram of topological spaces

$$\begin{array}{ccc} \tilde{X}_1 & \underset{f_{21}}{\overset{f_{12}}{\rightleftarrows}} & \tilde{X}_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ X_1 & \underset{\cong}{\overset{h}{\rightleftarrows}} & X_2 \end{array}$$

where p_i are the universal abelian covering maps of X_i and f_{12} and f_{21} are the liftings of $h \circ p_1$ and $h^{-1} \circ p_2$. f_{12} and f_{21} are well-defined, since $h_*(p_{1,*})(\pi_1(\tilde{X}_1)) = h_*([\pi_1(X_1), \pi_1(X_1)]) = [\pi_1(X_2), \pi_1(X_2)]$ and similarly for f_{21}. Furthermore $f_{12} \circ f_{21} \in \operatorname{Aut} \tilde{X}_2$ and $f_{21} \circ f_{12} \in \operatorname{Aut} \tilde{X}_1$. Thus we get induced group isomorphisms on homology

$$H_i(\tilde{X}_1) \underset{f_{21,*}}{\overset{f_{12,*}}{\rightleftarrows}} H_i(\tilde{X}_2)$$

We can choose τ_1 and τ_2 as the generators of $\operatorname{Aut}_{X_1} \tilde{X}_1$ and $\operatorname{Aut}_{X_2} \tilde{X}_2$ respectively such that the following diagram commutes

$$\begin{array}{ccc} H_i(\tilde{X}_1) & \underset{f_{21,*}}{\overset{f_{12,*}}{\rightleftarrows}} & H_i(\tilde{X}_2) \\ \downarrow{\scriptstyle \tau_{1,*}} & & \downarrow{\scriptstyle \tau_{2,*}} \\ H_i(\tilde{X}_1) & \underset{f_{21,*}}{\overset{f_{12,*}}{\rightleftarrows}} & H_i(\tilde{X}_2) \end{array}$$

It follows that $f_{12,*}$ and $f_{21,*}$ are module isomorphisms.

Therefore generators of $\operatorname{Aut}_{X_1}\tilde{X}_1$ and $\operatorname{Aut}_{X_2}\tilde{X}_2$ can be chosen so that $H_i(\tilde{X}_1)$ and $H_i(\tilde{X}_2)$ are isomorphic as Λ-module for each $i \in \mathbb{N}$. □

4.1.1 The Computation of $H_1(\tilde{X})$ using a Seifert surface

In this section it is assumed that $S^1 \approx K \subset S^3$. Thus a Seifert surface of K is a compact connected oriented surface. The following is a method for computing $H_1(\tilde{X})$ by using a Seifert surface of the knot.

Recall from Chapter 3 that given a knot $K \in S^3$ with complement X, $\tilde{X}$ can be constructed from copies of $Y = S^3 - M$ and $\varphi : \mathring{M} \times (-1, 1) \hookrightarrow S^3$, the open bicollar of $\mathring{M}$, where M is a Seifert surface with genus g. By the classification of compact connected oriented surfaces, we can find generators $a_1, ..., a_{2g}$ of $H_1(M)$ such that $H_1(M)$ is free abelian with basis $a_1, ..., a_{2g}$. Using the Alexander duality and the universal coefficient theorem, there are generators $\alpha_1, ..., \alpha_{2g}$ of $H_1(Y)$ such that $H_1(Y)$ is free abelian with basis $\alpha_1, ..., \alpha_{2g}$. Restrict φ to $\mathring{M} \times \{-\frac{1}{2}\}$ and consider the induced homomorphism $\varphi_*^- : H_1(\mathring{M} \times \{-\frac{1}{2}\}) \cong H_1(M) \to H_1(Y)$. Define $a_i^- := \varphi_*^-(a_i)$, $i = 1, 2, ..., 2g$. Similarly we restrict φ to $\mathring{M} \times \{\frac{1}{2}\}$ and define $a_i^+ := \varphi_*^+(a_i)$, $i = 1, 2, ..., 2g$. Thus a_i^- and a_i^+ can be written as linear combinations of $\alpha_1, ..., \alpha_{2g}$. Then we obtain a Λ-module presentation of $H_1(\tilde{X})$ as follows:

Proposition 4.3. *Fixing the generator τ_+ of $\operatorname{Aut}_X \tilde{X}$ that maps Y_i to Y_{i+1} and N_i to N_{i+1}, we have a Λ-module structure of $H_1(\tilde{X})$ with respect to τ_+. Then $H_1(\tilde{X})$ has a Λ-module presentation as follows*

$$H_1(\tilde{X}) \cong \langle \{\alpha_i\}_{i=1,2,...,2g} \mid a_j^- = t a_j^+, j = 1, 2, ..., 2g \rangle$$

where $\{\alpha_i\}_{i=1,2,...,2g}$ are the generators of $H_1(Y_0) \cong H_1(Y)$, $\{a_i\}_{i=1,2,...,2g}$ are the generators of $H_1(N_0) \cong H_1(M)$ and $a_i^- := N_{0*}^-(a_i)$, $a_i^+ := N_{0*}^+(a_i)$, a_i^-, $a_i^+ \in Y_0$, $i = 1, 2, ..., 2g$.

Proof. Consider a curve $\Gamma : \mathbb{R} \to \tilde{X}$ which is the lift of a meridian of K under the universal abelian cover. Consider the open covers of $\tilde{X}$ via

$$Y' = (\sqcup_{i \in \mathbb{Z}} Y_i) \cup U(\Gamma)$$

$$N' = (\sqcup_{i \in \mathbb{Z}} N_i) \cup U(\Gamma)$$

where $U(\Gamma)$ is a small open neighbourhood of Γ that strongly deformation retracts to Γ.

We have

$$Y' \cap N' \simeq \sqcup_{i\in\mathbb{Z}}(\varphi_i^+(\mathring{M} \times \{-\frac{1}{2}\}) \cup \varphi_i^-(\mathring{M} \times \{\frac{1}{2}\})) \cup U(\Gamma).$$

Since Y', N', $Y' \cap N'$ and $\tilde{X}$ are connected, there is the following Mayer-Vietoris sequence of reduced homology

$$... \to H_1(N' \cap Y') \xrightarrow{f} H_1(N') \oplus H_1(Y') \to H_1(\tilde{X}) \to 0 \ .$$

Therefore $H_1(\tilde{X})$ is isomorphic to the cokernel of f. From the Mayer-Vietoris sequence of the obvious covers of Y', N' and $Y' \cap N'$ we obtain

$$H_1(N' \cap Y') \cong \oplus_{i\in\mathbb{Z}}(H_1(\varphi_i^+(\mathring{M} \times \{-\frac{1}{2}\})) \oplus H_1(\varphi_i^-(\mathring{M} \times \{\frac{1}{2}\})))$$

$$H_1(N') \cong \oplus_{i\in\mathbb{Z}} H_1(N_i)$$

$$H_1(Y') \cong \oplus_{i\in\mathbb{Z}} H_1(Y_i) \ .$$

These groups are free Λ-modules with basis:

$$H_1(N' \cap Y') \cong \ \langle \{t^k a_j^+\}_{\substack{k\in\mathbb{Z} \\ j\in 1,2,...,2g}}, \ \{t^k a_j^-\}_{\substack{k\in\mathbb{Z} \\ j\in 1,2,...,2g}} \rangle$$

$$H_1(N') \cong \ \langle \{t^k a_j\}_{\substack{k\in\mathbb{Z} \\ j\in 1,2,...,2g}} \rangle$$

$$H_1(Y') \cong \ \langle \{t^k \alpha_j\}_{\substack{k\in\mathbb{Z} \\ j\in 1,2,...,2g}} \rangle \ .$$

Furthermore f is the map induced from the inclusions of $Y' \cap N'$ into Y' and N'. So

$$f(t^k a_j^+) = (t^k a_j, t^k a_j^+)$$

$$f(t^k a_j^-) = (t^k a_j, t^{k-1} a_j^-)$$

for $k \in \mathbb{Z}$, $j \in 1, 2, ..., 2g$.

Note that in $H_1(N') \oplus H_1(Y')$

$$(t^k a_j, 0) + (0, t^k a_j^+) = f(t^k a_j^+)$$

$$(t^k a_j, 0) + (0, t^k a_j^-) = f(t^{k-1} a_j^-)$$

for $k \in \mathbb{Z}$, $j \in 1, 2, ..., 2g$.

Therefore in coker f,

$$\left[(t^k a_j, 0)\right] = -\left[(0, t^k a_j^+)\right]$$

$$\big[(t^k a_j, 0)\big] = -\big[(0, t^{k-1} a_j^-)\big]$$

where $\big[\ \big]$ denotes the equivalence class in coker f. Thus we have

$$\begin{aligned} H_1(\tilde{X}) &\cong H_1(Y')/(t^k a_j^- \sim t^{k+1} a_j^+)_{\substack{k\in\mathbb{Z} \\ j\in 1,2,\ldots,2g}} \\ &\cong \ \langle t^k \alpha_j \mid t^{k-1} a_j^- = t^k a_j^+, k \in \mathbb{Z}, j \in \{1,2,\ldots,2g\}\rangle \end{aligned}$$

a group presentation of $H_1(\tilde{X})$, which leads to the Λ-module presentation of $H_1(\tilde{X})$

$$H_1(\tilde{X}) \cong \ \langle \{\alpha_j\} \mid a_j^- = t a_j^+,\ j \in \{1,2,\ldots,2g\}\rangle\ .$$

□

4.1.2 The Computation of $H_1(\tilde{X})$ from $\pi_1(X)$

Let $K \subset S^{n+2}$ be a knot with complement X and the universal abelian cover $p : \tilde{X} \to X$. Recall that $p_* : \pi_1(\tilde{X}) \to \pi_1(X)$ is an isomorphism onto the commutator subgroup $C = [G,G]$ where $G = \pi_1(X)$. Thus there is an induced group isomorphism:

$$\bar{p_*} : H_1(\tilde{X}) \to C/\big[C,C\big]\ .$$

This section describes how to define a Λ-module structure on $C/[C,C]$ in a purely algebraic manner which is compatible with the Λ-module structure on $H_1(\tilde{X})$ and makes the above a Λ-isomorphism. This leads to a method for computing a Λ-module presentation for $H_1(\tilde{X})$, given a presentation for the knot group.

Definition 4.4. Suppose $c \in C$. Let $x \in G$ be such an element that $p_G(x) = \pm 1$ where p_G denotes the abelianization $p_G : G \to G/C \cong \mathbb{Z}$. Define a group automorphism of $C/[C,C]$ via

$$t[c] = [xcx^{-1}]$$

where [ ] denotes the coset in $C/\big[C,C\big]$.

Lemma 4.5. *$t : C/\big[C,C\big] \to C/\big[C,C\big]$ is well defined.*

Proof. If $c,d \in C$ are congruent mod $\big[C,C\big]$ and $x,y \in G$ which are sent to generators of G/C are congruent mod $[G,G]$. Then $p_G(xcx^{-1}) = p_G(x)p_G(c)p_G(x^{-1}) = 0$ and similarly $p_G(ydy^{-1}) = 0$. Thus xcx^{-1} and ydy^{-1} are elements of C. Furthermore $(xcx^{-1})(ydy^{-1})^{-1} \in \big[C,C\big]$. To see this, we shall set $xy^{-1} = \tilde{c} \in C$ and $cd^{-1} = \bar{c} \in \big[C,C\big]$.

Then consider the image of $(xcx^{-1})(ydy^{-1})^{-1}$ under the natural projection $p_C : C \to C/[C,C]$

$$\begin{aligned} p_C((xcx^{-1})(ydy^{-1})^{-1}) &= p_C((xcx^{-1})\tilde{c}^{-1}xc^{-1}\bar{c}x^{-1}\tilde{c}) \\ &= p_C(xcx^{-1}) + p_C(\tilde{c}^{-1}) + p_C(xc^{-1}x^{-1}) + p_C(\tilde{c}) \\ &= p_C((xcx^{-1})\tilde{c}(xcx^{-1})^{-1}\tilde{c}^{-1}) \\ &= 0 \in C/[C,C] \ . \end{aligned}$$

□

Proposition 4.6. *The following diagram commutes*

$$\begin{array}{ccc} H_1(\tilde{X}) & \xrightarrow{\bar{p}_*} & C/[C,C] \\ \downarrow{\scriptstyle \tau_*} & & \downarrow{\scriptstyle t} \\ H_1(\tilde{X}) & \xrightarrow{\bar{p}_*} & C/[C,C] \end{array}$$

if a proper choice of generator of G/C is made in the definition of t.

Proof. Without loss of generality we choose the generator $\tau : \tilde{X} \to \tilde{X}$ of $\operatorname{Aut} \tilde{X}$ to be τ_+. Thus we have the following commutative diagram

$$\begin{array}{ccc} \pi_1(\tilde{X}, y_i) & \xrightarrow{p_*} & C = [\pi_1(X, p(y_i)), \pi_1(X, p(y_i))] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ \pi_1(\tilde{X}, y_{i+1}) & \xrightarrow{p_*} & C = [\pi_1(X, p(y_{i+1})), \pi_1(X, p(y_{i+1}))] \end{array}$$

where $u(\alpha) := \gamma^{-1}\alpha\gamma$ and $v(\beta) := (p_*(\gamma))^{-1}\beta(p_*(\gamma))$ for $\alpha \in \pi_1(\tilde{X}, y_i)$ and $\beta \in C$ with basepoint $p(y_i) = p(y_{i+1})$. γ is an equivalence class of paths in $\tilde{X}$ from y_{i+1} to y_i with $y_{i+1} \in Y_{i+1}$, $y_i \in Y_i$ and $\tau(y_i) = y_{i+1}$.

Abelianize the above diagram and choose $x = p(\gamma)$ which is sent to a generator under the abelianization p_G. Thus τ_* is the induced map from u and t is the induced map from v. □

It follows that the natural Λ-module structure on $C/[C,C]$ coming from the automorphism t makes $\bar{p}_*$ a Λ-isomorphism.

Now we are to introduce a method for computing a Λ-module presentation for $H_1(\tilde{X})$ from a group presentation of $\pi_1(X)$.

Proposition 4.7. *If a knot group $G = \pi_1(X)$ is finitely presentable, then it has a presentation of the form:*

$$G \cong \langle x, a_1, ..., a_p \mid r_1, ...r_q \rangle$$

where $p_G(x) = 1$ *and* $p_G(a_i) = 0$.

Proof. Assume $G \cong \langle y_1, ..., y_p \mid r_1, ..., r_q \rangle$. Then $\exists k_1, ...k_p \in \mathbb{Z}$, such that $p_G(y_i) = k_i$. Choose $x \in G$, such that $p_G(x) = 1$ and define $a_i = x^{k_i} y_i^{-1}$. Thus $p_G(a_i) = 0$. Then $G \cong \langle x, a_1, ...a_p \mid r'_1, ..., r'_q \rangle$ with the relations reformulated as linear combinations of $x, a_1, ...a_p$. □

Proposition 4.8. *Using the above notation, the commutator subgroup C of G is generated by all words of the form:*

$$x^k a_i^{\pm 1} x^{-k} \ .$$

Proof. Any element $c \in C$ can be written as

$$c = w_1 x^{k_1} w_2 x^{k_2} ... w_r x^{k_r} w_{r+1} \ .$$

where w_i is a word made up from $\{a_1, ...a_p\}$, i.e.

$$w_i = a_{i_1}^{m_1} a_{i_2}^{m_2} ... a_{i_{p_i}}^{k_{p_i}}, \ i \in \{1, 2, ..., r+1\}$$

.

Note that for any $n \in \mathbb{Z}$,

$$\begin{aligned} w_i[n] &:= x^n w_i x^{-n} \\ &= \prod_{k=1}^{p_i} (x^n a_{i_k} x^{-n})^{m_k} \end{aligned} \tag{4.1}$$

is of the desired form in the proposition.

Thus c can be rewritten as:

$$c = w_1 \left(\prod_{i=1}^{r} w_{i+1} [\sum_{j=1}^{i} k_j] \right) x^{\sum_{j=1}^{r} k_j}$$

Note that $p_G(c) = 0$ and $p_G(w_i) = 0$. So $\sum_{j=1}^{r} k_j = 0$. Therefore

$$c = w_1 \left(\prod_{i=1}^{r} w_{i+1} [\sum_{j=1}^{i} k_j] \right)$$

is generated by all words of the form $x^k a_i^{\pm 1} x^{-k}$. □

Furthermore, each r_i is equivalent to a word $r_i^{'}$ which is a product of words of that form, since $p_G(r_i) = 0$, i.e. $r_i \in C$.

Therefore we may obtain a Λ-module presentation of $C/[C,C]$ by taking generators $\alpha_1,...\alpha_p$, which are the images of $a_1,...,a_p$ under abelianization p_G. Formally rewriting r_i' additively, substituting

$$\pm t^k \alpha_i \text{ for } x^k a_i^{\pm 1} x^{-k} \ .$$

4.1.3 Example: The Alexander invariant of the torus knots

Before introducing the definition of the Alexander polynomial, we first give a concrete example of computation of $H_1(\tilde{X})$ with the method introduced above and thus in the next chapter we can obtain the Alexander polynomial from the presentation of $H_1(\tilde{X})$ as a Λ-module.

Proposition 4.9. *For a torus knot $T_{p,q}$ with p,q coprime positive integers, its Alexander invariant is $\Lambda/(\Delta_{p,q}(t))$ with $\Delta_{p,q}(t) = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)}$.*

Remark 4.10. In chapter 4.3 we will see that $\Delta_{p,q}(t)$ is the Alexander polynomial of $T_{p,q}$.

Proof. Recall from Chapter 2.2 that the knot group of a torus knot $T_{p,q}$ with p,q coprime positive integers has a presentation

$$G_{p,q} \cong \langle u, v | \ u^p = v^q \rangle \ .$$

Lemma 4.11. *Under the abelianization $p_G(u) = q,\ p_G(v) = p$.*

Proof. Assume $p_G(u) = k,\ p_G(v) = m$. From the relation $u^p = v^q$ we obtain $kp = mq$ and thus we can write $k = tq,\ m = tp,$ for some $t \in \mathbb{Z}$. Since u,v generates $G_{p,q}$, there exists $r,s \in \mathbb{Z}$ such that $rk + sm = 1$. Substituting k with tq, m with tp, we obtain $t(rq + sp) = 1$ and therefore $t = \pm 1$. If $t = -1$, take $G_{p,q} \cong \langle -u, -v \mid (-u)^p = (-v)^q \rangle$. Therefore without loss of generality we can assume that $p_G(u) = q,\ p_G(v) = p$. □

Since p,q are coprime, we can choose integers r,s satisfying $pr + qs = 1,\ r > 0,\ s < 0$. Let

$$x = u^s v^r,\ a = ux^{-q},\ b = vx^{-p} \ .$$

In this way we obtain a presentation of $G_{p,q}$

$$G_{p,q} \cong \langle x, a, b \mid (ax^q)^p = (bx^p)^q,\ x = (ax^q)^s (bx^p)^r \rangle$$

with $p_G(x) = 1,\ p_G(a) = 0,\ p_G(b) = 0$.

Rewrite the relation $(ax^q)^p = (bx^p)^q$ as

$$\prod_{i=0}^{p-1} x^{iq}ax^{-iq} = (\prod_{j=0}^{q-1} x^{jq}bx^{-jq}) \ .$$

Now we substitute $x^k ax^{-k}$ with $t^k\alpha$ and $x^k bx^{-k}$ with $t^k\beta$. Thus we obtain

$$(1 + t^q + t^{2q} + \dots + t^{(p-1)q})\alpha = (1 + t^p + t^{2p} + \dots + t^{(q-1)p})\beta \ . \tag{4.2}$$

Similarly the relation $x = (ax^q)^s(bx^p)^r$ can be substituted with

$$(1 + t^q + t^{2q} + \dots + t^{(-s-1)q})\alpha = (1 + t^p + t^{2p} + \dots + t^{(r-1)p})\beta \ . \tag{4.3}$$

The equation (4.1) can be rewritten as

$$\frac{1 - t^{pq}}{1 - t^q}\alpha = \frac{1 - t^{pq}}{1 - t^p}\beta$$

and the equation (4.2) can be rewritten as

$$\frac{1 - t^{-sq}}{1 - t^q}\alpha = \frac{1 - t^{rp}}{1 - t^p}\beta \ .$$

Thus we have a Λ-module presentation of the Alexander invariant of the torus knot $T_{p,q}$

$$H_1(\tilde{X}_{p,q}) = \left\langle \alpha, \beta \ \middle| \ \frac{1 - t^{pq}}{1 - t^q}\alpha = \frac{1 - t^{pq}}{1 - t^p}\beta, \ \frac{1 - t^{-sq}}{1 - t^q}\alpha = \frac{1 - t^{rp}}{1 - t^p}\beta \right\rangle \ .$$

Define Λ-module homomorphisms $\phi : H_1(\tilde{X}_{p,q}) \to \Gamma = < \gamma \mid \Delta(t)\gamma = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)}\gamma = 0 >$ via

$$\phi(\alpha) = \frac{1 - t^{rp}}{1 - t^p}\gamma$$

$$\phi(\beta) = \frac{1 - t^{-sq}}{1 - t^q}\gamma.$$

and $\psi : \Gamma \to H_1(\tilde{X}_{p,q})$ via

$$\psi(\gamma) = t^{sq}\left(\frac{1 - t^p}{1 - t}\alpha - \frac{1 - t^q}{1 - t}\beta\right) \ .$$

First we check that ϕ and ψ are well-defined,i.e.

$$\frac{1 - t^{pq}}{1 - t^q}\phi(\alpha) - \frac{1 - t^{pq}}{1 - t^p}\phi(\beta) = \frac{\Delta(t)}{t^{sq}}\gamma = 0$$

$$\frac{1-t^{-sq}}{1-t^q}\phi(\alpha) - \frac{1-t^{rp}}{1-t^p}\phi(\beta) = 0\cdot\gamma = 0$$
$$\Delta(t)\psi(\gamma) = t^{sq}\left(\frac{1-t^{pq}}{1-t^q}\alpha - \frac{1-t^{pq}}{1-t^p}\beta\right) = 0\ .$$

Second we check that ϕ and ψ are inverse to each other where we only need to check this on the generators

$$\begin{aligned}
\phi\circ\psi(\gamma) &= \phi\left(t^{sq}\left(\frac{1-t^p}{1-t}\alpha - \frac{1-t^q}{1-t}\beta\right)\right)\\
&= t^{sq}\frac{1-t^p}{1-t}\phi(\alpha) - t^{sq}\frac{1-t^q}{1-t}\phi(\beta)\\
&= t^{sq}\left(\frac{1-t^p}{1-t}\frac{1-t^{rp}}{1-t^p} - \frac{1-t^q}{1-t}\frac{1-t^{-sq}}{1-t^q}\right)\gamma\\
&= t^{sq}\frac{t^{-sq}(1-t^{qs+pr})}{1-t}\gamma\\
&= \gamma
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
\psi\circ\phi(\alpha) &= \psi\left(\frac{1-t^{rp}}{1-t^p}\gamma\right)\\
&= \frac{1-t^{rp}}{1-t^p}\left(t^{sq}\left(\frac{1-t^p}{1-t}\alpha - \frac{1-t^q}{1-t}\beta\right)\right)\\
&= t^{sq}\left(\frac{1-t^{rp}}{1-t}\alpha - \frac{1-t^q}{1-t}\frac{1-t^{rp}}{1-t^p}\beta\right)\\
&\overset{(4.2)}{=} t^{sq}\left(\frac{1-t^{rp}}{1-t}\alpha - \frac{1-t^q}{1-t}\frac{1-t^{-sq}}{1-t^q}\alpha\right)\\
&= t^{sq}\frac{t^{-sq}(1-t^{qs+pr})}{1-t}\alpha\\
&= \alpha
\end{aligned} \tag{4.5}$$

since $qs+pr=1$.

Similarly we have $\psi\circ\phi(\beta)=\beta$.

Thus $\psi\circ\phi = id_{H_1(\tilde{X}_{p,q})}$ and $\phi\circ\psi = id_\Gamma$. So we obtain $H_1(\tilde{X}_{p,q}) \cong \Gamma$. Furthermore $\Gamma \cong \Lambda/\Delta_{p,q}(t)$. Therefore we concluded that the Alexander invariant of a torus knot $T_{p,q}$ is $\Lambda/\Delta_{p,q}(t)$. □

4.2 Presentation of modules

In this section we introduce the matrix presentation of a module and later apply this to the computation of the Alexander polynomial.

Let A be a commutative ring with unit and consider a finitely presented module M over A:

$$M \cong \langle \alpha_1, ...\alpha_r \mid \rho_1, ..., \rho_s \rangle$$

where each relation $\rho_i, i = 1, 2, ..., s$ is a linear combination of the generators:

$$\rho_i = \Sigma_{j=1}^{r} a_{ij}\alpha_j, a_{ij} \in A, \ i = 1, 2, ..., p \ .$$

Define $P = (a_{ij})_{s\times r}$ as the *presentation matrix* for M corresponding to the given module presentation. That is, the rows of P are the coefficients of the relators relative to the generators. Knowing P is the same as knowing the specific presentation to which it corresponds. Therefore P determines M up to A-isomorphism.

Definition 4.12. Let M be a module over A which has an $s \times r$ presentation matrix P. The ideal of A generated by all $r \times r$ minors is called the *order ideal* of M. If $s < r$, it is defined to be the zero ideal.

Proposition 4.13. *The order ideal of an A-module M does not depend on the choice of presentation P.*

For a proof of the Proposition 4.13, see [Zas58, Chapter III].

Remark 4.14. In the case where M has a square presentation matrix P, the order ideal is principal and is generated by $\det P$.

4.3 The Alexander polynomial

Recall that the Alexander invariant $H_*(\tilde{X})$ is the homology groups of the universal abelian cover of the knot complement, considered as a module over Λ, the ring of integral Laurent polynomials. In the case of classical tame knots it is finitely presentable (we can compute the so-called Wirtinger presentation from a regular projection of the knot [Rol76, Chapter 3.D]), and it follows from the Mayer-Vietoris sequence in Proposition 4.3 that $H_i(\tilde{X})$ vanish for $i \geq 2$.

Definition 4.15. Any presentation matrix for the Alexander invariant $H_1(\tilde{X})$ of a knot K is called an *Alexander matrix* for K. The associated order ideal in Λ is called the *Alexander ideal* of K, and if this is principal, any generator of the Alexander ideal is called the *Alexander polynomial.*

Proposition 4.16. *The Alexander polynomial of a torus knot $T_{p,q}$ is* $\Delta_{p,q}(t) = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)}$.

Proof. Recall from Proposition 4.9 a presentation of the Alexander invariant $H_1(\tilde{X}_{p,q})$ of the torus knot $T_{p,q}$

$$H_1(\tilde{X}_{p,q}) \cong \Lambda \big/ (\Delta_{p,q}(t)) \cong \langle \gamma \mid \Delta_{p,q}(t)\gamma = 0 \rangle$$

with $\Delta_{p,q}(t) = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)}$.

It follows that the presentation matrix of $H_1(\tilde{X}_{p,q})$ is a 1×1 matrix with $\Delta_{p,q}(t)$ as its entry. Therefore the Alexander polynomial of the torus knot $T_{p,q}$ is

$$\Delta_{p,q}(t) = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)} .$$

□

Now we can give a proof of the Schreier's theorem 2.12 using the Alexander polynomial of the torus knot.

Proposition 4.17. *Let $p > 1$ and $q > 1$. Then the polynomials $\Delta_{p_1,q_1}(t)$ and $\Delta_{p_2,q_2}(t)$ for T_{p_1,q_1} and T_{p_2,q_2} with $p_i, q_i \in \mathbb{N}_+,\ i = \{1,2\}$ are distinct unless $\{p_1, q_1\} = \{p_2, q_2\}$.*

Proof. The roots of $\Delta_{p,q}(t)$ are all the pq-th roots of unity except for the p-th roots of unity and q-th roots of unity. The proposition follows by comparing the sets of roots of $\Delta_{p_1,q_1}(t)$ and $\Delta_{p_2,q_2}(t)$. □

From now on we restrict our attention to classical knots in S^3. Note especially that a tame knot in S^3 has a presentation of $H_1(\tilde{X})$ with as many generators as relations (e.g. Wirtinger presentation, see [Rol76, Chapter 3.D]). That is, for a given tame knot $K \subset S^3$, there is a square matrix presentation of $H_1\tilde{X}$ as a $\Lambda-$ module, and its Alexander polynomial is the determinant of the corresponding matrix.

4.3.1 Seifert form and Seifert matrices

One of the many ways to compute the Alexander polynomial is by using Seifert matrices, where we need first the notion of the linking number of two knots.

Definition 4.18. Given J and K two disjoint oriented knots in S^3 (or $\mathbb{R}^3$), let M be a Seifert surface for K, with bicollar $(N,\ N^+,\ N^-)$ of $\mathring{M}$ as in Chapter 3. Assume (allowing adjustment of J by a homotopy in $S^3 - K$) that J meets M at a finite number of points, and at each such point J passes locally

(a) from N^+ to N^-

(b) from N^- to N^+

following its orientation. Weight the intersections of type (a) with +1 and those of type (b) with −1. The sum of these numbers is the *linking number* of J and K, denoted by $\mathrm{lk}\,(J,K)$.

Remark 4.19. Although it is not obvious from the definition, the linking number does not depend of the choice of Seifert surface, see [Rol76, Chapter 5.D].

Given a knot K in S^3, choose a Seifert surface M in S^3 for K and also choose a particular bicollar $\mathring{M} \times [-1,1]$ in $S^3 - K$. If $x \in H_1(\mathring{M})$ is represented by a closed curve (which we'll also call x) in $\mathring{M}$, let x^+ denote the closed curve carried by $x \times \{1\}$ in the bicollar. Similarly let x^- denote the closed curve carried by $x \times \{-1\}$. Since disjoint closed curves in S^3 have a well-defined linking number we can make the following definition.

Definition 4.20. The function $f : H_1(\mathring{M}) \times H_1(\mathring{M}) \to \mathbb{Z}$ defined by $(x,y) \mapsto lk(x,y^+)$ is called a *Seifert form* for K. It clearly depends upon the choice of M and choice of a bicollar. We choose further a basis $e_1,...e_{2g}$ (M is of genus g) for $H_1(\mathring{M})$ as a $\mathbb{Z}$–module and define the associated *Seifert matrix* $V = (v_{ij})$ to be the $2g$ by $2g$ integral matrix with entries

$$v_{ij} = \mathrm{lk}\,(e_i, e_j^+)\ .$$

For a tame knot K in S^3 we have the following important theorem:

Theorem 4.21. *If V is a Seifert matrix for a tame knot K in S^3, then $V^T - tV$ is an Alexander matrix for K. So is $V - tV^T$.*

For the proof of the theorem, we need first the following lemma:

Lemma 4.22. *Let M be a Seifert surface for a knot in S^3 and let $a_1,...a_{2g}$ be a basis for $H_1(\mathring{M})$. Then there is a basis $\alpha_1,...,\alpha_{2g}$ for $H_1(S^3 - M)$ which is dual to $\{a_i\}$ with respect to the linking pairing. That is,* $\mathrm{lk}\,(a_i,\alpha_j) = \delta_{ij}$, $i,j \in \{1,2,...,2g\}$.

Proof. See [Rol76, Chapter 8.C.14]. □

Now we are to prove the theorem:

Proof. Let M be a bicollared Seifert surface for K and let V be the Seifert matrix corresponding to some basis $a_1,...,a_{2g}$ for $H_1(\mathring{M})$. Its entries are $v_{ij} = \mathrm{lk}\,(a_i,a_j^+)$. By the lemma, let $\alpha_1,\alpha_2,...,\alpha_{2g}$ be the basis for $H_1(S^3 - M)$ which is dual to $\{a_i\}$ with respect to the linking pairing, i.e. $\mathrm{lk}\,(a_i,\alpha_j) = \delta_{ij}$. Note that the coefficients of any element $\alpha = \Sigma c_j \alpha_j$ may be recovered by $c_j = \text{lk}(\alpha, a_j)$. As in the section 4.1, we see that the homology $H_1(\tilde{X})$ can be presented as a Λ-module with generators $\alpha_1, ..., \alpha_{2g}$ and relations

$$a_i^- = ta_i^+, \; i = 1, ..., 2g \; .$$

Writing out the relations in terms of the α_j they becomes

$$\sum_j \text{lk}(a_i^-, a_j)\alpha_j = t\left(\sum_j \text{lk}(a_i^+, a_j)\alpha_j\right) \; .$$

From the definition we have that $\text{lk}(a_i^-, a_j) = \text{lk}(a_i, a_j^+) = v_{ij}$, the relations may be rewriten as

$$\sum (v_{ij} - tv_{ij})\alpha_j = 0, \; i = 1, ..., 2g \; .$$

The relation matrix corresponding to this is precisely $V - tV^T$. Note that if one interchanges the + and − sides of the bicollar of M, the new Seifert matrix is just the transpose of the old one. From this we can conclude that $V^T - tV$ is also a presentation matrix for $H_1(\tilde{X})$. □

Corollary 4.23. *The Alexander invariant of a tame knot in S^3 has a square presentation matrix $V^T - tV$, so its Alexander ideal is principal and it has Alexander polynomial $\Delta(t) := \det(V^T - tV)$.*

Corollary 4.24. *Define the* degree *of a Laurent polynomial to be the difference between the highest and lowest exponents at which nonzero coefficients occur. Then there is the following inequality connecting the genus of a knot in S^3 and the degree of its Alexander polynomial:*

$$\deg(\Delta(t)) \leq 2g(K)$$

Proof. Assume we use a minimal Seifert surface of genus $g = g(K)$ to compute the Serfiert matrix. Then the highest degree of t in the Alexander polynomial is $2g$. Thus $\deg(\Delta(t)) \leq 2g(K))$. □

Corollary 4.25. *The genus of the torus knot $T_{p,q}$ is $\frac{1}{2}(p-1)(q-1)$.*

Proof. Recall from chapter 3.1 that we concluded that $g(T_{p,q}) \leq \frac{(p-1)(q-1)}{2}$. In chapter 4.1.3 we computed that the Alexander polynomial of $T_{p,q}$ is a polynomial of degree $(p-1)(q-1)$. Thus from the corollary above $g(T_{p,q}) \geq \frac{(p-1)(q-1)}{2}$. Therefore $g(T_{p,q}) = \frac{(p-1)(q-1)}{2}$. □

4.3.2 Example: The Alexander invariant of the twist knots

Given a handle decomposition of a compact connected oriented genus one surface M with boundary, recall that the twist knot $K_{m,n}$, $m, n \in \mathbb{Z}$ is defined as the boundary of M after fully twisting left and right 1-handles m and n times respectively.

We can choose M to be a Seifert surface of $K_{m,n}$, $g(M) = 1$.

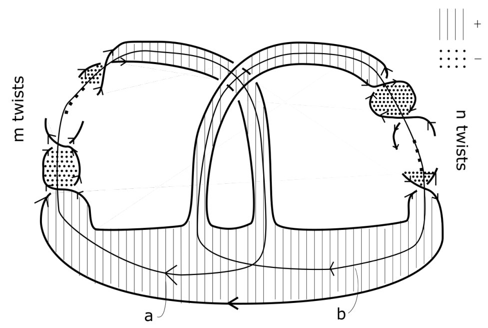
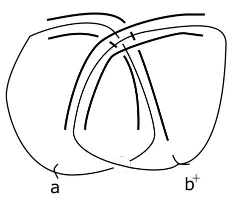


FIGURE 4.1: $K_{m,n}$ and M considered as its Seifert surface

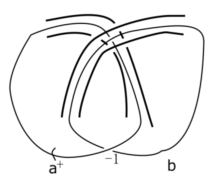
We shall fix an orientation of the knot as shown in Figure 4.1 to determine the + and − sides of the bicollar of the surface M. Choose the generators a and b of M and fix their orientation as indicated in Figure 4.1, then

$$V(K_{m,n}) = \begin{bmatrix} \operatorname{lk}(a, a^+) & \operatorname{lk}(a, b^+) \\ \operatorname{lk}(b, a^+) & \operatorname{lk}(b, b^+) \end{bmatrix}$$

First we observe that $\operatorname{lk}(a, b^+)$ and $\operatorname{lk}(b, a^+)$ do not depend on the pair $(m, n) \in \mathbb{Z}$, since they can only be linked with each other at the place where the two 1-handles over/under-cross each other. In particular, $\operatorname{lk}(a, b^+) = 0$ and $\operatorname{lk}(b, a^+) = -1$.



(A) a lies under the Seifert surface of b^+



(B) b passing from the - side to + side of the Seifert surface of a^+

As for $\operatorname{lk}(a, a^+)$, the linking takes place at the m full twists of the left 1-handle. If $m > 0$, a passes from the + side to the - side of the Seifert surface of a^+ m times; if $m < 0$, a passes from the - side to the + side of the Seifert surface of a^+ $-m$ times. Thus

$\mathrm{lk}(a, a^+) = m$. Similarly for $\mathrm{lk}(b, b^+)$ which depends on the n ful twists of the right 1-hanlde, we obtain $lk(b, b^+) = n$. Therefore we obtain the Seifert matrix of the knot $K_{m,n}$ as follows

$$V(K_{m,n}) = \begin{bmatrix} m & 0 \\ -1 & n \end{bmatrix}.$$

From the Seifert matrix we can calculate the Alexander polynomial $\Delta_{m,n}(t)$ of $K_{m,n}$

$$\Delta_{m,n}(t) = mnt^2 + (1 - 2mn)t + mn \ .$$

Proposition 4.26. *If $mn \neq 0$, the genus of the knot $K_{m,n}$ is 1.*

Proof. Since M is a Seifert surface of $K_{m,n}$, $g(K) \leq g(M) = 1$. The Alexander polynomial of $K_{m,n}$ has degree 2 if $mn \neq 0$. Thus by Corollary 4.24 $g(K) \geq 1$. Therefore we obtain $g(K) = 1$. □

Example 4.1. *In the case where $n = 0$, we obtain a Seifert surface M_0 with $m+4$ Seifert circles and $m+5$ crossings by applying the Seifert algorithm. Using Proposition 3.11 we have $g(M_0) = 0$. Therefore $K_{m,0}$ is the unknot. Similarly when $m = 0$ we get $K_{0,n}$ is the unknot.*

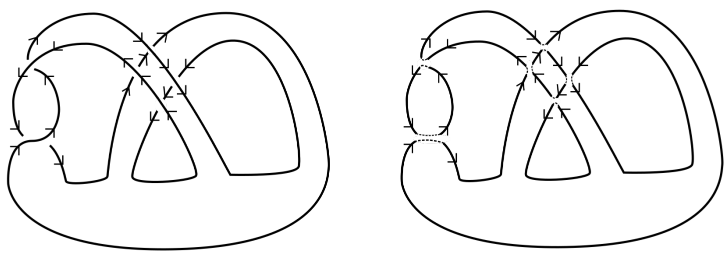


FIGURE 4.3: $K_{2,0}$ and the Seifert circles obtained by the Seifert algorithm

Proposition 4.27. *The Alexander polynomial of the unknot is $\Delta_u(t) = t$.*

Proof. Let $m = 0$, then the Alexander polynomial of $K_{0,n}$ is $\Delta_{0,n}(t) = t$ and $K_{0,n}$ is the unknot. □

Example 4.2. *$K_{1,1}$ is a right-handed trefoil knot and $K_{-1,-1}$ is a left-handed trefoil knot.*

We show by pictures how $K_{1,1}$ can be transformed to a right-handed trefoil using isotopies and similarly it works for $K_{-1,-1}$

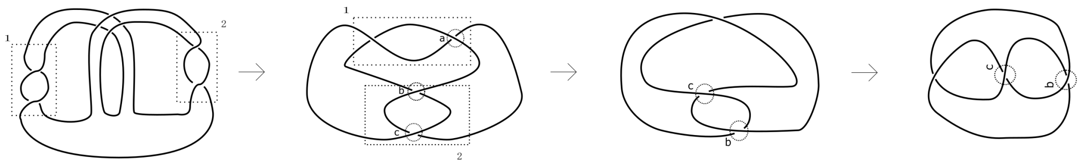


FIGURE 4.4: $K_{1,1}$ is a right-handed trefoil knot

Proposition 4.28. *A twist knot $K_{m,n}$ with $mn \neq 0$ can be embedded in a standard torus, i.e, it is also a torus knot if and only if $mn = 1$.*

Proof. From the above example we know that $K_{1,1}$ and $K_{-1,-1}$ are trefoil knot, i.e. $T_{2,3}$. As for the other direction, recall from Corollary 4.25 and Proposition 4.26 that $g(T_{p,q}) = \frac{(p-1)(q-1)}{2}$ and $g(K_{m,n}) = 1$. Furthermore $\frac{(p-1)(q-1)}{2} = 1$ exactly when $\{p,q\} = \{2,3\}$, i.e. $K_{m,n}$ is equivalent with the trefoil knot. Therefore $\Delta_{m,n}(t) = \frac{(1-t)(1-t^6)}{(1-t^2)(1-t^3)} = t^2 - t + 1$ and by comparing coefficients we obtain $mn = 1$. □

Bibliography

[Ada04] Colin Conrad Adams. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* American Mathematical Society, 2004.

[Deh14] Max Dehn. "Die beiden Kleeblattschlingen. (German)". In: *Math. Ann.* 75 (1914), pp. 402–413.

[GL89] Cameron Gordon and John Luecke. "Knots are determined by their complements." In: *J. Amer. Math. Soc.* 2 (1989), 371415.

[HH63] Richard H.Crowell and Ralph H.Fox. *Introduction to Knot Theory.* Springer New York, 1963.

[Mas77] William S. Massey. *Algebraic topology: An introduction.* Reading, Massachusetts: Springer-Verlag New York, 1977.

[Rol76] Dale Rolfsen. *Knots and Links.* Berkeley,CA: Publish or Perish, 1976.

[RS72] C. Rourke and B. Sanderson. *Introduction to piecewise-linear topology.* Berlin-Heidelberg -New York: Springer-Verlag, 1972.

[Sch24] Otto Schreer. "Über die Gruppen $A^a B^b = 1$. (German)". In: *Abh. Math. Sem. Univ. Hamburg* 3 (1924), pp. 167–169.

[W.H33] Morris W.Hirsch. *Differential Topology.* Spring-Verlag New York Inc., 1933.

[WC06] Jarke J.van Wijk and Arjeh M. Cohen. "Visualization of Seifert Surfaces." In: *IEEE Transactions on Visualization andComputer Graphics* 5 (2006).

[Zas58] Hans Zassenhaus. *The Theory of Groups.* Vandenhoeck and Ruprecht, 1958.